Chapter 1. Combinatorial Theory

1.2: More Counting

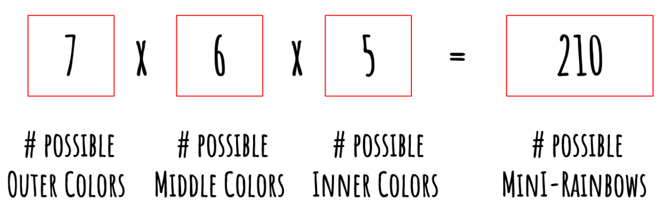
Slides (Google Drive) Video (YouTube)

1.2.1 k-Permutations

Last time, we learned the foundational techniques for counting (the sum and product rule), and the factorial notation which arises frequently. Now, we'll learn even more "shortcuts"/"notations" for common counting situations, and tackle more complex problems.

We'll start with a simpler situation than most of the exercises from last time. How many 3-color mini rainbows can be made out of 7 available colors, with all 3 being different colors?

We choose an outer color, then a middle color and then an inner color. There are 7 possibilities for the outer layer, 6 for the middle and 5 for the inner (since we cannot have duplicates). Since order matters, we find that the total number of possibilities is 210, from the following calculation:



Let's manipulate our equation a little and see what happens.

$$\begin{aligned} 7 \cdot 6 \cdot 5 &= \frac{7 \cdot 6 \cdot 5}{1} \cdot \frac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} && \text{[multiply numerator and denominator by } 4! = 4 \cdot 3 \cdot 2 \cdot 1\text{]} \\ &= \frac{7!}{4!} && \text{[def of factorial]} \\ &= \frac{7!}{(7-3)!} \end{aligned}$$

Notice that we are "picking" 3 out of 7 available colors - so order matters. This may not seem useful, but imagine if there were 835 colors and we wanted a rainbow with 135 different colors. You would have to multiply 135 numbers, rather than just three!

Definition 1.2.1: k-Permutations

If we want to arrange **only** k out of n distinct objects, the number of ways to do so is $P(n,k)$ (read as "n pick k"), where

$$P(n,k) = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

A **permutation** of a n objects is an arrangement of each object (where order matters), so a **k-permutation** is an arrangement of k members of a set of n members (where order matters). The number of k-Permutations of n objects is just $P(n,k)$.

Example(s)

Suppose we have 13 chairs (in a row) with 9 TA's, and Professors Sunny, Rainy, Windy, and Cloudy to be seated. What is the number of seatings where every professor has a TA to his/her immediate left *and* right?

Solution This is quite a tricky problem if we don't choose the right setup. Imagine we first just order 9 TA's in a line - there are 9! ways to do this. Then, there are 8 spots between them, so that if we place a professor there, they're guaranteed to have a TA to their immediate left and right. We can't place more than one professor in a spot. Out of the 8 spots, we **pick** 4 of them for the professors to sit (order matters, since the professors are different people). So the answer by the product rule is $9! \cdot P(8,4)$. □

1.2.2 k-Combinations (Binomial Coefficients)

What if order *doesn't* matter? For example, if I need to **choose** 3 out of 7 weapons on my online adventure? We'll tackle that now, continuing our rainbow example!

A kindergartener smears 3 different colors out of 7 to make a new color. How many smeared colors can she create?

Notice that there are $3! = 6$ possible ways to order red, blue and orange, as you see below. However, all these rainbows produce the same "smeared" color!

Recall that there were $P(7,3) = 210$ possible mini-rainbows. But as we see from these rainbows, each "smeared" color is counted $3! = 6$ times. So, to get our answer, we take the 210 mini-rainbows and divide by 6 to account for the overcounting since in this case, order doesn't matter.

The answer is,

$$\frac{210}{6} = \frac{P(7,3)}{3!} = \frac{7!}{3!(7-3)!}$$

Definition 1.2.2: k-Combinations (Binomial Coefficients)

If we want to choose (order doesn't matter) **only** k out of n distinct objects, the number of ways to do so is $C(n,k) = \binom{n}{k}$ (read as "n choose k"), where

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

A **k-combination** is a selection of k objects from a collection of n objects, in which the order does not matter. The number of k-Combinations of n objects is just $\binom{n}{k}$. $\binom{n}{k}$ is also called a **binomial coefficient** - we'll see why in the next section.

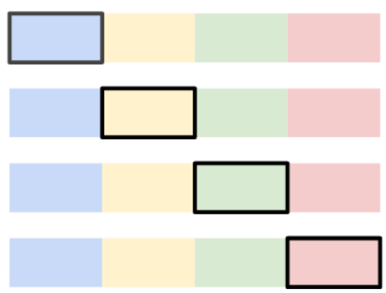
Notice, we can show from this that there is symmetry in the definition of binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$$

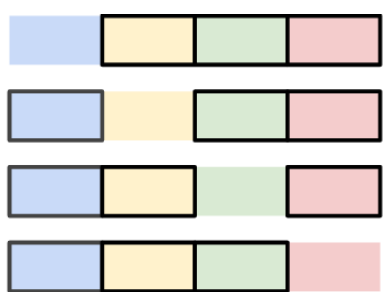
The algebra checks out - why is this true though, intuitively?

Let's suppose that $n = 4$ and $k = 1$. We want to show $\binom{4}{1} = \binom{4}{3}$. We have 4 colors:

These are the possible ways to choose 1 color out of 4:



These are the possible ways to choose 3 colors out of 4:



Looking at these, we can see that the color choices in each row are complementary. Intuitively, choosing 1 color is the same as choosing $4 - 1 = 3$ colors that we don't want - and vice versa. This explains the symmetry in binomial coefficients!

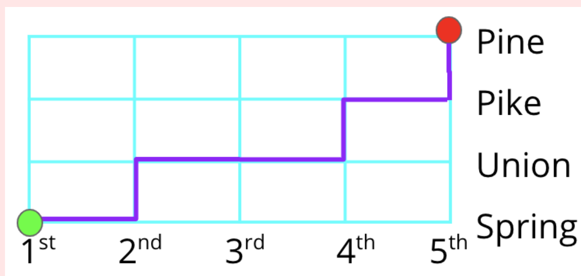
Example(s)

There are 6 AI professors and 7 theory professors taking part in an escape room. If 4 AI professors and 4 theory professors are to be chosen and divided into 4 pairs (one AI professor with one theory professor per pair), how many pairings are possible?

Solution We first *choose* 4 out of 6 AI professors, with order not mattering, and 4 out of 7 theory professors, again with order not mattering. There are $\binom{6}{4} \cdot \binom{7}{4}$ ways to do this by the product rule. Then, for the first theory professor, we have 4 choices of AI professor to match with, for the second theory professor, we only have 3 choices, and so on. So we multiply by 4! to pair them off, and we get $\binom{6}{4} \cdot \binom{7}{4} \cdot 4!$. You may have counted it differently, but check if your answer matches! □

Example(s)

How many ways are there to walk from the intersection of 1^{st} and Spring to 5^{th} and Pine? Assume we only go North and East. A sample route is highlighted in purple.



Solution We can actually solve this problem as well! It has a rather clever solution.

We have to move North exactly three times and East exactly four times. Let's encode a path as a sequence of 3 N's and 4 E's (the path highlighted is encoded as ENEENEN). Then, let's *choose* the three positions for the N's, giving us $\binom{7}{3}$ ways (why not *pick*?). Then, the E's are actually already determined right? They have to be in the remaining 4 positions. So the answer is simply $\binom{7}{3}$. Alternatively, if we wanted to choose the positions for the 4 N's first instead, there would be $\binom{7}{4}$ ways to do this.
Remember that $\binom{7}{3} = \binom{7}{7-3} = \binom{7}{4}$ so these are equivalent! □

1.2.3 Multinomial Coefficients

Now we'll see if we can generalize our binomial coefficients to solve even more interesting problems. Actually, they can be derived easily from binomial coefficients.

How many ways can you arrange the letters in "MATH"?

$4! = 24$, since they are distinct objects.

But if we want to rearrange the letters in "POOPOO", we have indistinct letters (two types - P and O). How do we approach this?

One approach is to choose where the 2 P's go, and then the O's have to go in the remaining 4 spots ($\binom{4}{4} = 1$ way) . Or, we can choose where the 4 O's go, and then the remaining P's are set ($\binom{2}{2} = 1$ way).

Either way, we get,

$$\binom{6}{2} \cdot \binom{4}{4} = \binom{6}{4} \cdot \binom{2}{2} = \frac{6!}{2!4!}$$

Another interpretation of this formula is that we are first arranging the 6 letters as if they were distinct: $P_1O_1O_2P_2O_3O_4$. Then, we divide by 4! and 2! to account for 4 duplicate O's and 2 duplicate P's.

What if we got even more complex, let's say three different letters? For example, rearranging the word "BABYYYBAY". There are 3 B's, 2 A's, and 4 Y's, for a total of 9 letters. We can choose where the 3 B's should go of the 9 spots: $\binom{9}{3}$ (order doesn't matter since all the B's are identical). Then out of the remaining 6 spots, we should choose 2 for the A's: $\binom{6}{2}$. Finally, out of the 4 remaining spots, we put the 4 Y's there: $\binom{4}{4} = 1$. By the product rule, our answer is

$$\binom{9}{3}\binom{6}{2}\binom{4}{4} = \frac{9!}{3!6!}\frac{6!}{2!4!}\frac{4!}{4!0!} = \frac{9!}{3!2!4!}$$

Note that we could have chosen to assign the Y's first instead: Out of 9 positions, we choose 4 to be Y: $\binom{9}{4}$. Then from the 5 remaining spots, choose where the 2 A's go: $\binom{5}{2}$, and the last three spots must be B's: $\binom{3}{3} = 1$. This gives us the equivalent answer

$$\binom{9}{4}\binom{5}{2}\binom{3}{3} = \frac{9!}{4!5!}\frac{5!}{2!3!}\frac{3!}{3!0!} = \frac{9!}{3!2!4!}$$

This shows once again that there are many correct ways to count something. This type of problem also frequently appears, and so we have a special notation (called a **multinomial coefficient**)

$$\binom{9}{3,2,4} = \frac{9!}{3!2!4!}$$

Note the order of the bottom three numbers does not matter (since the multiplication in the denominator is commutative), and that the bottom numbers must add up to the top number.

Definition 1.2.3: Multinomial Coefficients

If we have k types of objects (n total), with n_1 of the first type, n_2 of the second, ..., and n_k of the k-th, then the number of arrangements possible is

$$\binom{n}{n_1, n_2, ..., n_k} = \frac{n!}{n_1!n_2!...n_k!}$$

This is a **multinomial coefficient**, the generalization of binomial coefficients.

Above, we had $k = 3$ objects (B, A, Y) with $n_1 = 3$ (number of B's), $n_2 = 2$ (number of A's), and $n_3 = 4$ (number of Y's), for an answer of $\binom{n}{n_1,n_2,n_3} = \frac{9!}{3!2!4!}$.

Example(s)

How many ways can you arrange the letters in "GODOGGY"?

Solution There are $n = 7$ letters. There are only $k = 4$ distinct letters - $\{G, O, D, Y\}$.
$n_1 = 3$ - there are 3 G's.
$n_2 = 2$ - there are 2 O's.
$n_3 = 1$ - there is 1 D.
$n_4 = 1$ - there is 1 Y.
This gives us the number of possible arrangements:

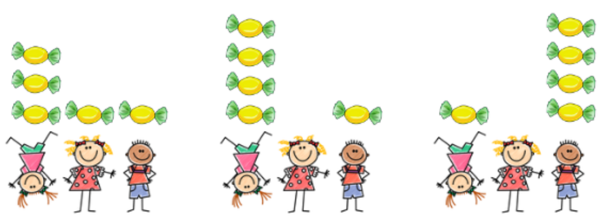
$$\binom{7}{3,2,1,1} = \frac{7!}{3!2!1!1!}$$

It is important to note that even though the 1's are "useless" since $1! = 1$, we still must write every number on the bottom since they have to add to the top number. □

1.2.4 Stars and Bars/Divider Method

Now we tackle another common type of problem, which seems complicated at first. It turns out though that it can be reduced to binomial coefficients!

How many ways can we give 5 (indistinguishable) candies to these 3 (distinguishable) kids? Here are three possible distributions of candy:



Notice that the second and third pictures show different possible distributions, since the kids are distinguishable (different). Any idea on how we can tackle this problem?

The idea here is that we will count something equivalent. Let's say there are 5 "stars" for the 5 candies and 2 "bars" for the dividers (dividing 3 kids). For instance, this distribution of candies corresponds to this arrangement of 5 stars and 2 bars:

Here is another example of the correspondence between a distribution of candies and the arrangement of stars and bars:

For each candy distribution, there is exactly one corresponding way to arrange the stars and bars. Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.

Hence, the number of ways to distribute 5 candies to the 3 kids is the number of arrangements of 5 stars and 2 bars.

This is simply

$$\binom{7}{2} = \binom{7}{5} = \frac{7!}{2!5!}$$

Amazing right? We just reduced this candy distribution problem to reordering letters!

Theorem 1.2.1: Stars and Bars/Divider Method

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+(k-1)}{k-1} = \binom{n+(k-1)}{n}$$

since we set up n stars for the n balls, and $k-1$ bars dividing the k bins.

Example(s)

There are 20 students and 4 professors. Assume the students are indistinguishable to the professors; who only care *how many* students they have, and not which ones.

1. If there are no restrictions, how many ways can we assign the students to the professors?

Solution This is actually the perfect setup for stars and bars. We have 20 stars (students) and 3 bars (professors), and so our answer is $\binom{23}{3} = \binom{23}{20}$. □

1.2.5 Exercises

1. There are 40 seats and 40 students in a classroom. Suppose that the front row contains 10 seats, and there are 5 students who must sit in the front row in order to see the board clearly. How many seating arrangements are possible with this restriction?

   Solution: Again, there may be many correct approaches. We can first choose which 5 out of 10 seats in the front row we want to give, so we have $\binom{10}{5}$ ways of doing this. Then, assign those 5 students to these seats, to which there are 5! ways. Finally, assign the other 35 students in any way, for 35! ways. By the product rule, there are $\binom{10}{5} \cdot 5! \cdot 35!$ ways to do so.

2. Suppose you are to get to take your final exam in pairs. There are 100 students in the class and 8 TAs, so 8 lucky students will get to pair up with a TA! Each TA must take the exam with some student, but two TAs cannot take the exam together. How many ways can they pair up?

   Solution: First we choose the 8 lucky students and pair them with a TA. There are $\binom{100}{8}$ ways to choose those 8 students and then 8! ways to pair them up, for a total of $\binom{100}{8} \cdot 8!$ ways (note this is the same as $P(100, 8)$). Then there are 92 students left. The first one has 91 choices. Then there are 90 students left, and so the next one has 89 choices. And so on. So the total number of ways is

   $$\binom{100}{8} \cdot 8! \cdot 91 \cdot 89 \cdot 87 \cdot \ldots 3 \cdot 1$$

3. If we roll a fair 3-sided die 11 times, what is the number of ways that we can get 4 1's, 5 2's, and 2 3's?

   Solution: We can write the outcomes as a sequence of length 11, each digit of which is 1, 2 or 3. Hence, the number of ways to get 4 1's, 5 2's, and 2 3's, is the number of orderings of 11112222233, which is $\binom{11}{4,5,2} = \frac{11!}{4!5!2!}$.

4. These two problems are almost identical, but have drastically different approaches to them. These are both extremely hard/tricky problems, though they may look deceivingly simple. These are probably the two coolest problems I've encountered in counting, as they do have elegant solutions!

   (a) How many 7-digit phone numbers are such that the numbers are strictly increasing (digits must go up)? (e.g., 014-5689, 134-6789, etc.)

   (b) How many 7-digit phone numbers are such that the numbers are monotone increasing (digits can stay the same or go up)? (e.g., 011-5566, 134-6789, etc.) Hint: Reduce this to stars and bars.

   Solution:

   (a) We choose 7 out of 10 digits, which has $\binom{10}{7}$ possibilities, and then once we do, there is only 1 valid ordering (must put them in increasing order). Hence, the answer is simply $\binom{10}{7}$. This question has a deceivingly simple solution, as many students (including myself at one point), would have started by choosing the first digit. But the choices for the next digit depend on the first digit. And so on for the third. This leads to a complicated, nearly unsolvable mess!

   (b) This is a very difficult problem to frame in terms of stars and bars. We need to map one phone number to exactly one ordering of stars and bars, and vice versa. Consider letting the 9 bars being an increase from one-digit to the next, and 7 stars for the 7 digits. This is extremely complicated, so we'll give 3 examples of what we mean.

      i. The phone number 011-5566 is represented as $*|**||||**|**|||$. We start a counter at 0, we see a digit first (a star), so we mark down 0. Then we see a bar, which tells us to increase our counter to 1. Then, two more digits (stars), which say to mark down 2 1's. Then, 4 bars which tell us to increase count from 1 to 5. Then two *'s for the next two 5's, and a bar to increase to 6. Then, two stars indicate to put down 2 6's. Then, we increment count to 9 but don't put down any more digits.

      ii. The phone number 134-6789 is represented as $|*||*|*||*|*|*|*$. We start a counter at 0, and we see a bar first, so we increase count to 1. Then a star tells us to actually write down

1 as our first digit. The two bars tell us to increase count from 1 to 3. The star says mark a 3 down now. Then, a bar to increase to 4. Then a star to write down 4. Two bars to increase to 6. And so on.

iii. The stars and bars ordering $|||| * | * * * *|| * ||*$ represents the phone number 455-5579. We start a counter at 0. We see 4 bars so we increment to 4. The star says to mark down a 4. Then increment count by 1 to 5 due to the next bar. Then, mark 5 down 4 times (4 stars). Then increment count by 2, put down a 7, and repeat to put down a 9.

Hence there is a bijection between these phone numbers and arrangements of 7 stars and 9 bars. So the number of satisfying phone numbers is $\binom{16}{7} = \binom{16}{9}$.