Probability

9.8 Multi-Armed bandits

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Slides by Alex Tsun

Agenda

- The Multi-Armed Bandit (MAB) Problem
- Greedy/Epsilon-Greedy
- Upper Confidence Bound (UCB)
- Thompson Sampling
- Modern Hypothesis Testing

Multi-Armed Bandit (MAB) Problem

- K Slot Machines {1,2,...,K} (aka "Bandits" with "Arms").

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- <u>**Goal**</u>: Maximize total (expected) reward after T time steps.

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- **Goal**: Maximize total (expected) reward after T time steps.

- **Problem**: At each time step, decide which arm to pull based on past history of rewards.

Multi-Armed Bandit (MAB) Problem

Below has the reward distribution of each of the K=3 arms.

What's your strategy to maximize your total (expected) reward?

$Poi(\lambda = 1.36)$

$Bin(n = 10, p = 0.4)$

$\mathcal{N}(\mu = -1, \sigma^2 = 4)$

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Pull arm 2 every time since it has the highest expected reward!

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This is a hard problem - we know nothing about the K reward distributions!

Multi-Armed Bandit (MAB) Problem

Need to balance the tradeoff between:

Exploitation: Pulling arm(s) we know to be "good".
Exploration: Pulling other arms in the hopes they are also "good" or even better.

Bernoulli Bandits

We will handle the case of Bernoulli-bandits. That is, reward of arm a $\in$ {1,2,…,K} is Ber(p_a) .

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$Ber(p_2)$

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Observe: The expected reward of arm a is just p_a (expectation of Bernoulli).

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We don't know these!

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Want Avg-Regret(T) → 0 as T → ∞ . **Minimizing Regret = Maximizing Reward.**

(Bernoulli) Bandit Framework

How do we choose an arm at each time step (depending on past history), to maximize our total reward?

Algorithm 1 (Bernoulli) Bandit Framework

1: Have K arms, where pulling arm $i \in \{1, \ldots, K\}$ gives $Ber(p_i)$ reward ▷ p_i's all unknown.

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This is the focus of the rest of this lecture!

Motivation: Clinical Trials

K = 4 Arms (Treatments)

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Maximize: Total number of patients healed.

Motivation: Recommending Movies

K Movies

For visitor t, recommend movie $a_t \in \{1,2,\ldots,K\}$.

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For visitor t, recommend movie $a_t \in \{1,2,\ldots,K\}$.

Observe reward $r_t \in \{1,2,3,4,5\}$. (rating)

Maximize: Total/average rating of recommendations.

Motivation: Real Life?? (Food)

K Cuisines/Dishes (a ton)

For meal t, eat dish $a_t \in \{1,2,\ldots,K\}$.

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Maximize: Total/average happiness :)

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Observe reward $r_t \in \{1,2,3,4,5\}$. (happiness rating)

Maximize: Total/average happiness :)

The Question of the Day: Explore or Exploit????

Motivation: Real Life?? (Activities)

K Activities

On day t, do activity $a_t \in \{1,2,\ldots,K\}$.

Observe reward $r_t \in \{1,2,3,4,5\}$. (happiness rating)

Maximize: Total/average happiness :)

The Question of the Day: Explore or Exploit????

Any Ideas on what strategy we can use???

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    Estimate $\hat{p}_i = \dfrac{\sum_{j=1}^{M} r_{ij}}{M}$.

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for $t = KM + 1, KM + 2, \ldots, T$ **do**:
    Pull arm $a_t = a^*$. ▷ Pull the same arm for the rest of time.
    Receive reward $r_t \sim Ber(p_{a_t})$.

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Choose a number of times M to pull each arm initially, with KM ≤ T.
for i = 1, 2, ..., K do
    Pull arm i M times, observing iid rewards r_{i1}, ..., r_{iM} ~ Ber(p_i).
    Estimate p̂_i = (Σ_{j=1}^{M} r_{ij}) / M.
Determine best (empirical) arm a* = argmax_{i∈{1,2,...,K}} p̂_i.    ▹ We could be wrong...
for t = KM + 1, KM + 2, ..., T do:
    Pull arm a_t = a*.                         ▹ Pull the same arm for the rest of time.
    Receive reward r_t ~ Ber(p_{a_t}).
```

If we make a mistake, we will regret our decision for the rest of time....

Can we not do all of our exploration at the beginning?

Epsilon-Greedy Algorithm

ε

Explore with probability epsilon!

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for $t = KM + 1, KM + 2, \ldots, T$ **do**:
    if $Ber(\varepsilon) == 1$: **then** ▹ With probability ε, explore.
        Pull arm $a_t = Unif(1, K)$ (discrete). ▹ Choose a uniformly random arm.

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        Pull arm a_t = Unif(1, K) (discrete).     ▷ Choose a uniformly random arm.
    else                                          ▷ With probability 1 − ε, exploit.
        Pull arm a_t = arg max_{i∈{1,2,...,K}} p̂_i.  ▷ Choose arm with highest estimated reward.
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Can we explore more “naturally”?

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This algorithm constructs confidence intervals for the estimates of each arm, and chooses the arm with the highest **upper** confidence bound (if the confidence interval is [a,b], we compare only the value of b)

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for $i = 1, 2, \ldots, K$ **do**
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for $t = K+1, K+2, \ldots, T$ **do**:
    Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.

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Takes the upper part of of
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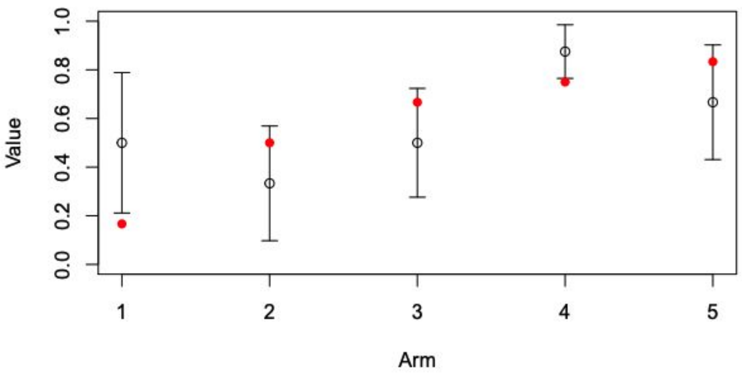
Exploration is "baked in": the frequently pulled arms will have narrow confidence intervals (and hence a lower **upper** bound), and the less-frequently pulled arms will have wide intervals (and hence a **higher** upper bound).

UCB: Confidence Intervals over Time

Confidence Intervals for Mean of Each Arm: t=10



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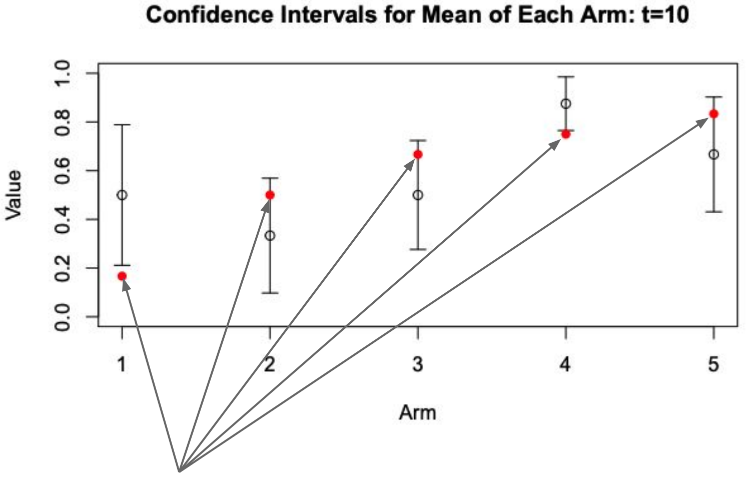
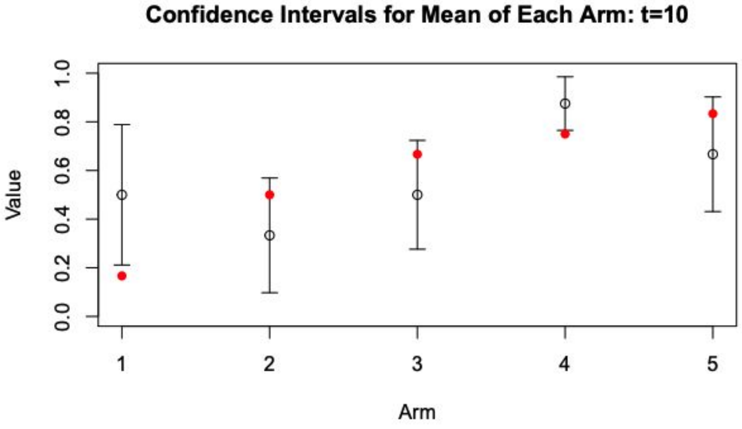
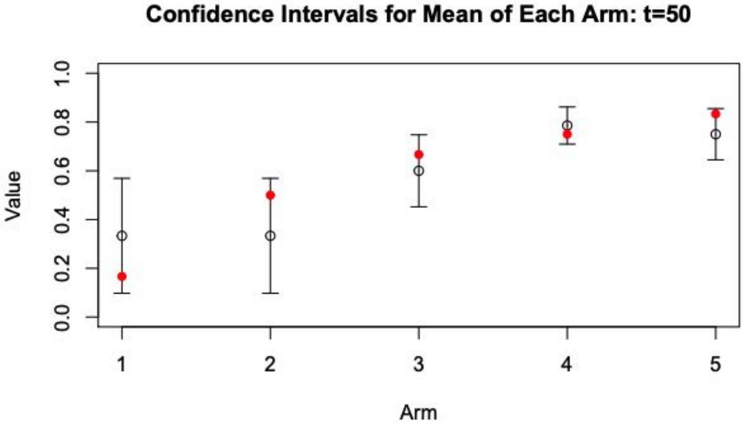


Red Dots: True Means

UCB: Confidence Intervals over Time





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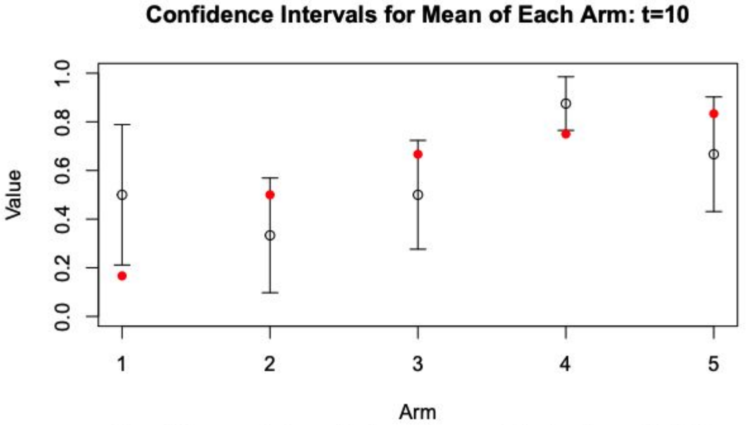

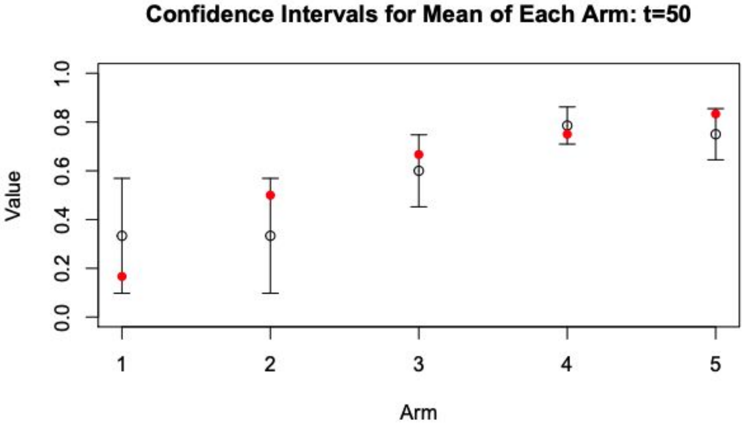

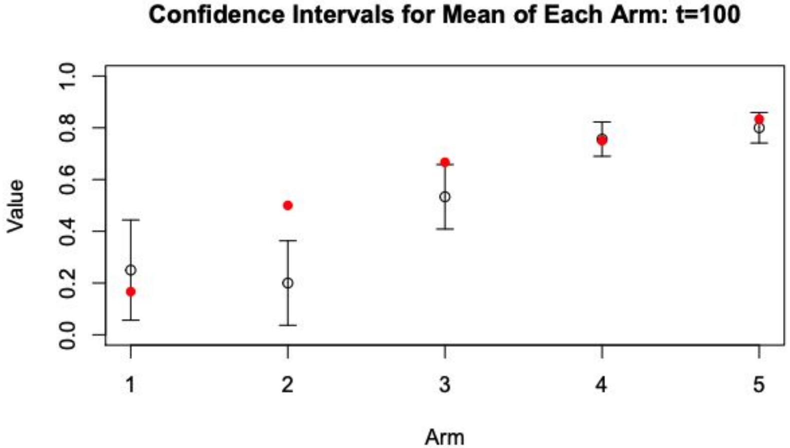

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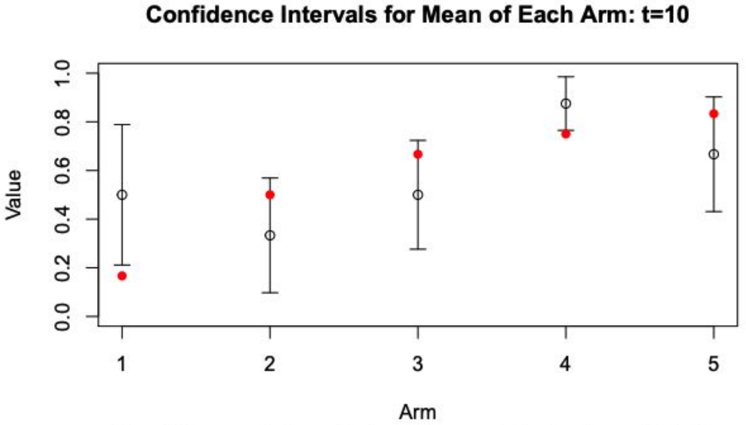

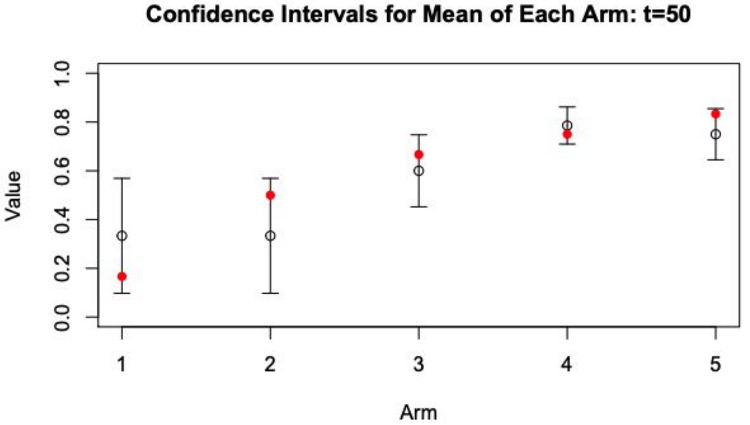

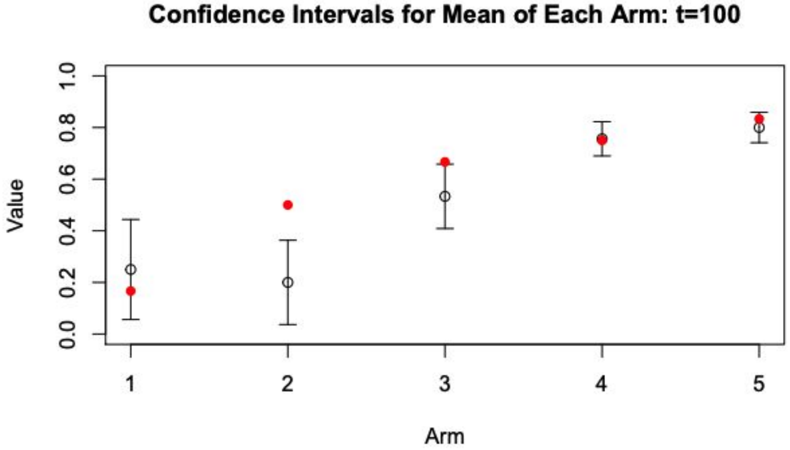

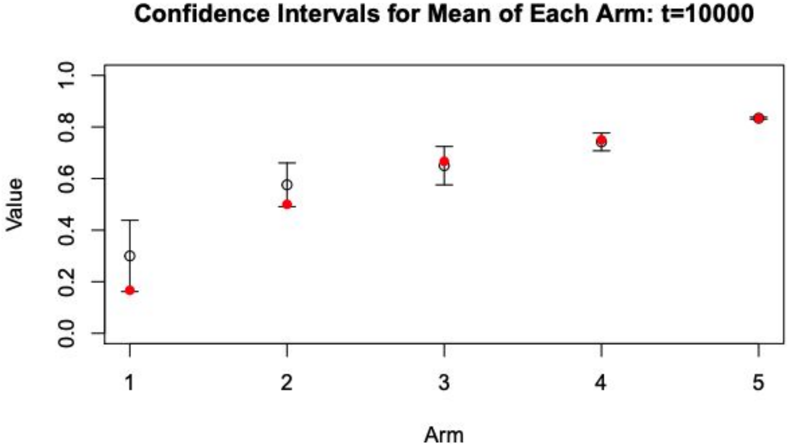

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \dots, K$ **do**
    Pull arm i once, observing $r_i \sim Ber(p_i)$.
    Estimate $\hat{p}_i = r_i$. ▹ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \dots, T$ **do:**
    Pull arm $a_t = \arg\max_{i \in \{1,2,\dots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
    Receive reward $r_t \sim Ber(p_{a_t})$.
    Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5				
2	0.2				
3	0.9				

Time (t)	Arm Pulled (a_t)	Reward (r_t)

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
  Pull arm i once, observing $r_i \sim Ber(p_i)$.
  Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
  Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}\right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
  Receive reward $r_t \sim Ber(p_{a_t})$.
  Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5				
2	0.2				
3	0.9				

Time (t)	Arm Pulled (a_t)	Reward (r_t)

We don't actually know these...

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
    Pull arm i once, observing $r_i \sim Ber(p_i)$.
    Estimate $\hat{p}_i = r_i$.   ▹ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do:**
    Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}\right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
    Receive reward $r_t \sim Ber(p_{a_t})$.
    Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5				
2	0.2				
3	0.9				

Time (t)	Arm Pulled (a_t)	Reward (r_t)
1	1	0

At time 1, we pull arm 1, and observe either a 1 (with probability 0.5) or a 0 (with probability 1-0.5). We happen to observe a 0.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
    Pull arm i once, observing $r_i \sim Ber(p_i)$.
    Estimate $\hat{p}_i = r_i$.   ▹ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
    Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
    Receive reward $r_t \sim Ber(p_{a_t})$.
    Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	
2	0.2				
3	0.9				

Time (t)	Arm Pulled (a_t)	Reward (r_t)
1	1	0

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
    Pull arm i once, observing $r_i \sim Ber(p_i)$.
    Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do:**
    Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
    Receive reward $r_t \sim Ber(p_{a_t})$.
    Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	
2	0.2				
3	0.9				

Time (t)	Arm Pulled (a_t)	Reward (r_t)
2	2	0

At time 2, we pull arm 2, and observe either a 1 (with probability 0.2) or a 0 (with probability 1-0.2).
We happen to observe a 0.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
    Pull arm i once, observing $r_i \sim Ber(p_i)$.
    Estimate $\hat{p}_i = r_i$.   ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
    Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
    Receive reward $r_t \sim Ber(p_{a_t})$.
    Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	
2	0.2	1	0	0/1	
3	0.9				

Time (t)	Arm Pulled (a_t)	Reward (r_t)
2	2	0

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
- Pull arm i once, observing $r_i \sim Ber(p_i)$.
- Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.

for $t = K+1, K+2, \ldots, T$ **do**:
- Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
- Receive reward $r_t \sim Ber(p_{a_t})$.
- Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	
2	0.2	1	0	0/1	
3	0.9				

Time (t)	Arm Pulled (a_t)	Reward (r_t)
3	3	1

At time 3, we pull arm 3, and observe either a 1 (with probability 0.9) or a 0 (with probability 1-0.9). We happen to observe a 1.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
    Pull arm i once, observing $r_i \sim Ber(p_i)$.
    Estimate $\hat{p}_i = r_i$.   ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
    Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
    Receive reward $r_t \sim Ber(p_{a_t})$.
    Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	
2	0.2	1	0	0/1	
3	0.9	1	1	1/1	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
3	3	1

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
    Pull arm i once, observing $r_i \sim Ber(p_i)$.
    Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do:**
    Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
    Receive reward $r_t \sim Ber(p_{a_t})$.
    Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	
2	0.2	1	0	0/1	
3	0.9	1	1	1/1	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
4		

At time 4, we must compute all our upper confidence bounds, and choose the best one.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
  Pull arm i once, observing $r_i \sim Ber(p_i)$.
  Estimate $\hat{p}_i = r_i$. ▹ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
  Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
  Receive reward $r_t \sim Ber(p_{a_t})$.
  Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	1.665
2	0.2	1	0	0/1	
3	0.9	1	1	1/1	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
4		

$$0 + \sqrt{\frac{2\ln(4)}{1}} \approx 1.665$$

At time 4, we must compute all our upper confidence bounds, and choose the best one.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
  Pull arm i once, observing $r_i \sim Ber(p_i)$.
  Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
  Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}\right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
  Receive reward $r_t \sim Ber(p_{a_t})$.
  Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	1.665
2	0.2	1	0	0/1	1.665
3	0.9	1	1	1/1	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
4		

$$0 + \sqrt{\frac{2\ln(4)}{1}} \approx 1.665$$

At time 4, we must compute all our upper confidence bounds, and choose the best one.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

```
for i = 1, 2, ..., K do
    Pull arm i once, observing r_i ~ Ber(p_i).
    Estimate p̂_i = r_i.                      ▹ Each estimate p̂_i will initially either be 1 or 0.
for t = K + 1, K + 2, ..., T do:
    Pull arm a_t = arg max_{i∈{1,2,...,K}} ( p̂_i + sqrt( 2 ln(t) / N_t(i) ) ), where N_t(i) is the number of times arm i was pulled before time t.
    Receive reward r_t ~ Ber(p_{a_t}).
    Update N_t(a_t) and p̂_{a_t} (using newly observed reward r_t).
```

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	1.665
2	0.2	1	0	0/1	1.665
3	0.9	1	1	1/1	2.665

Time (t)	Arm Pulled (a_t)	Reward (r_t)
4		

$$1 + \sqrt{\frac{2\ln(4)}{1}} \approx 2.665$$

At time 4, we must compute all our upper confidence bounds, and choose the best one.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
  Pull arm i once, observing $r_i \sim Ber(p_i)$.
  Estimate $\hat{p}_i = r_i$. ▹ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
  Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}\right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
  Receive reward $r_t \sim Ber(p_{a_t})$.
  Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	1.665
2	0.2	1	0	0/1	1.665
3	0.9	1	1	1/1	2.665

Time (t)	Arm Pulled (a_t)	Reward (r_t)
4	3	

At time 4, arm 3 has the highest UCB so we pull it.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

1: **for** $i = 1, 2, \ldots, K$ **do**
2: Pull arm i once, observing $r_i \sim Ber(p_i)$.
3: Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
4: **for** $t = K+1, K+2, \ldots, T$ **do**:
5: Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
6: Receive reward $r_t \sim Ber(p_{a_t})$.
7: Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	1.665
2	0.2	1	0	0/1	1.665
3	0.9	1	1	1/1	2.665

Time (t)	Arm Pulled (a_t)	Reward (r_t)
4	3	0

At time 4, arm 3 has the highest UCB so we pull it. We observe a reward of 0.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

1: **for** $i = 1, 2, \ldots, K$ **do**
2: Pull arm i once, observing $r_i \sim Ber(p_i)$.
3: Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
4: **for** $t = K+1, K+2, \ldots, T$ **do**:
5: Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}\right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
6: Receive reward $r_t \sim Ber(p_{a_t})$.
7: Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	
2	0.2	1	0	0/1	
3	0.9	2	1	1/2	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
4	3	0

At time 4, arm 3 has the highest UCB so we pull it. We observe a reward of 0. Then we update our estimate for p_3.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
  Pull arm i once, observing $r_i \sim Ber(p_i)$.
  Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K + 1, K + 2, \ldots, T$ **do:**
  Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
  Receive reward $r_t \sim Ber(p_{a_t})$.
  Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	
2	0.2	1	0	0/1	
3	0.9	2	1	1/2	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
5		

At time 5, we must compute all our upper confidence bounds, and choose the best one.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
    Pull arm i once, observing $r_i \sim Ber(p_i)$.
    Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
    Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}\right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
    Receive reward $r_t \sim Ber(p_{a_t})$.
    Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	1.794
2	0.2	1	0	0/1	
3	0.9	2	1	1/2	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
5		

$$0 + \sqrt{\frac{2\ln(5)}{1}} \approx 1.794$$

At time 5, we must compute all our upper confidence bounds, and choose the best one.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
  Pull arm i once, observing $r_i \sim Ber(p_i)$.
  Estimate $\hat{p}_i = r_i$. ▷ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
  Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
  Receive reward $r_t \sim Ber(p_{a_t})$.
  Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	1.794
2	0.2	1	0	0/1	1.794
3	0.9	2	1	1/2	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
5		

$$0 + \sqrt{\frac{2\ln(5)}{1}} \approx 1.794$$

At time 5, we must compute all our upper confidence bounds, and choose the best one.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
  Pull arm i once, observing $r_i \sim Ber(p_i)$.
  Estimate $\hat{p}_i = r_i$. ▹ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
  Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}} \right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
  Receive reward $r_t \sim Ber(p_{a_t})$.
  Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	**UCB** ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	1.794
2	0.2	1	0	0/1	1.794
3	0.9	2	1	1/2	1.769

Time (t)	Arm Pulled (a_t)	Reward (r_t)
5		

$$\frac{1}{2} + \sqrt{\frac{2\ln(5)}{2}} \approx 1.769$$

At time 5, we must compute all our upper confidence bounds, and choose the best one.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

for $i = 1, 2, \ldots, K$ **do**
  Pull arm i once, observing $r_i \sim Ber(p_i)$.
  Estimate $\hat{p}_i = r_i$. ▹ Each estimate $\hat{p}_i$ will initially either be 1 or 0.
for $t = K+1, K+2, \ldots, T$ **do**:
  Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} \left(\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}\right)$, where $N_t(i)$ is the number of times arm i was pulled before time t.
  Receive reward $r_t \sim Ber(p_{a_t})$.
  Update $N_t(a_t)$ and $\hat{p}_{a_t}$ (using newly observed reward r_t).

Arm (i)	True p_i	# Times Pulled	Total Reward	$\hat{p}_i$	UCB ($\hat{p}_i + \sqrt{\frac{2\ln(t)}{N_t(i)}}$)
1	0.5	1	0	0/1	1.794
2	0.2	1	0	0/1	1.794
3	0.9	2	1	1/2	1.769

Time (t)	Arm Pulled (a_t)	Reward (r_t)
5	1	

At time 5, arms 1 and 2 have the highest UCB so we pull one of them (let's break ties by choosing the smaller index arm). So we pull arm 1.

UCB Example

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Time (t)	Arm Pulled (a_t)	Reward (r_t)
5	1	0

At time 5, arms 1 and 2 have the highest UCB so we pull one of them (let's break ties by choosing the smaller index arm). So we pull arm 1.
We observe a reward of 0.

UCB Example

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Time (t)	Arm Pulled (a_t)	Reward (r_t)
5	1	0

At time 5, arms 1 and 2 have the highest UCB so we pull one of them (let's break ties by choosing the smaller index arm). So we pull arm 1.
We observe a reward of 0. Then we update our estimate for p_1.

UCB Example

Algorithm 4 UCB1 Algorithm (Upper Confidence Bound) for Bernoulli Bandits

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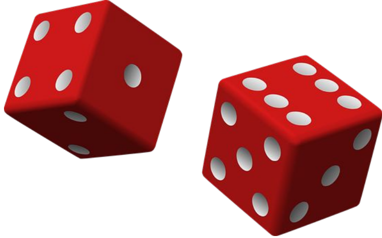
3	0.9	2	1	1/2	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
6		

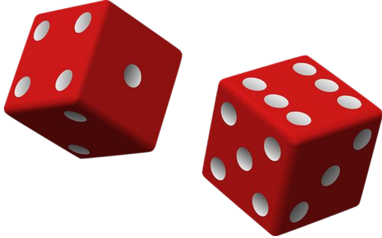
And so on!!! Notice how we started exploring since the confidence bound grows with t for even the unexplored arms!

Thompson Sampling Algorithm



Use MAP: Assume a Beta(1,1) (Uniform) prior on each unknown probability of reward.

Thompson Sampling Algorithm



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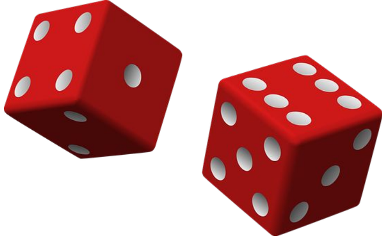
Algorithm 5 Thompson Sampling Algorithm for Beta-Bernoulli Bandits

1: For each arm $i \in \{1, \ldots, K\}$, initialize $\alpha_i = \beta_i = 1$. ▷ Set $Beta(\alpha_i, \beta_i)$ prior for each p_i.

Thompson Sampling Algorithm



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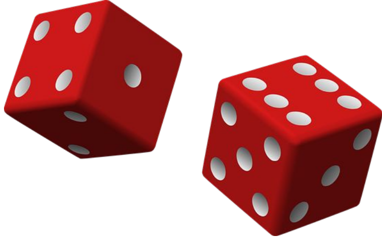
Algorithm 5 Thompson Sampling Algorithm for Beta-Bernoulli Bandits

For each arm $i \in \{1, \ldots, K\}$, initialize $\alpha_i = \beta_i = 1$. ▹ Set $Beta(\alpha_i, \beta_i)$ prior for each p_i.
for $t = 1, 2, \ldots, T$ **do**:
    For each arm i, get sample $s_{i,t} \sim Beta(\alpha_i, \beta_i)$. ▹ Each is a float in $[0, 1]$.

Thompson Sampling Algorithm



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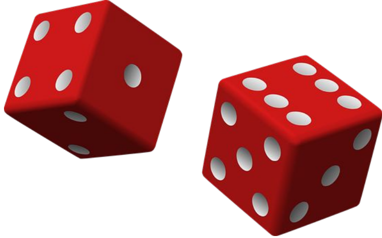
    For each arm i, get sample $s_{i,t} \sim Beta(\alpha_i, \beta_i)$. ▷ Each is a float in $[0, 1]$.

    Pull arm $a_t = \arg\max_{i \in \{1,2,\ldots,K\}} s_{i,t}$. ▷ This "bakes in" exploration!

Thompson Sampling Algorithm



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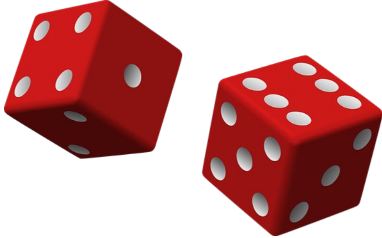
    Receive reward $r_t \sim Ber(p_{a_t})$.

Thompson Sampling Algorithm



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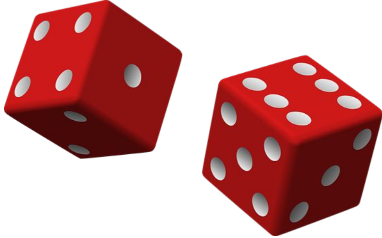
Thompson Sampling Algorithm



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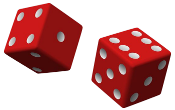
The exploration comes in since we **sample** from each Beta distribution, rather than just choosing the one with largest expectation or mode (greedy).

Thompson Example



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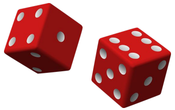
Arm (i)	True p_i	α_i	β_i	$s_{i,t}$
1	0.5			
2	0.2			
3	0.9			

Time (t)	Arm Pulled (a_t)	Reward (r_t)

Thompson Example



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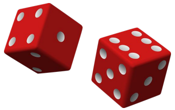
Arm (i)	True p_i	α_i	β_i	$s_{i,t}$
1	0.5	1	1	
2	0.2	1	1	
3	0.9	1	1	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
1		

Thompson Example



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```
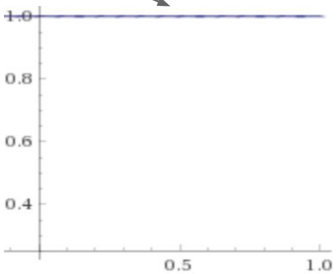
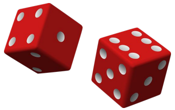

Arm (i)	True p_i	α_i	β_i	$s_{i,t}$
1	0.5	1	1	0.43
2	0.2	1	1	
3	0.9	1	1	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
1		

Sample from Beta(1,1) density →



Thompson Example



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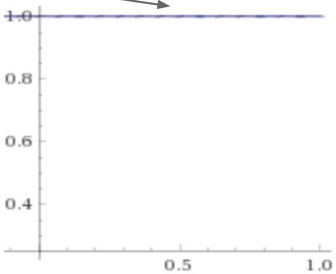
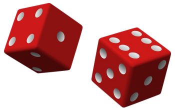
Arm (i)	True p_i	α_i	β_i	$s_{i,t}$
1	0.5	1	1	0.43
2	0.2	1	1	0.75
3	0.9	1	1	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
1		

Sample from Beta(1,1) density →



Thompson Example



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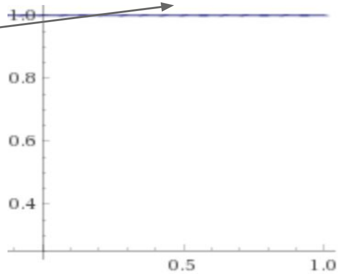
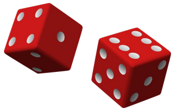
1	0.5	1	1	0.43
2	0.2	1	1	0.75
3	0.9	1	1	0.11

Sample from Beta(1,1) density →



Time (t)	Arm Pulled (a_t)	Reward (r_t)
1		

Thompson Example



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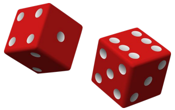
Time (t)	Arm Pulled (a_t)	Reward (r_t)
1	2	

Choose arm with highest sample!

Thompson Example



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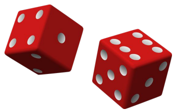
Time (t)	Arm Pulled (a_t)	Reward (r_t)
1	2	0

Observe reward 1 with probability 0.2 and 0 with probability 0.8.

Thompson Example



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Time (t)	Arm Pulled (a_t)	Reward (r_t)
1	2	0

Add a count of 1 to the failures :(.

Thompson Example

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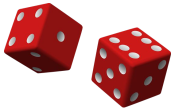
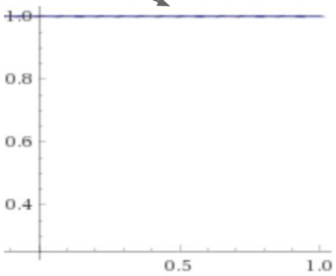


Arm (i)	True p_i	α_i	β_i	$s_{i,t}$
1	0.5	1	1	0.52
2	0.2	1	2	
3	0.9	1	1	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
2		

Sample from Beta(1,1) density →



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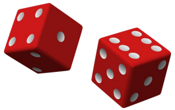

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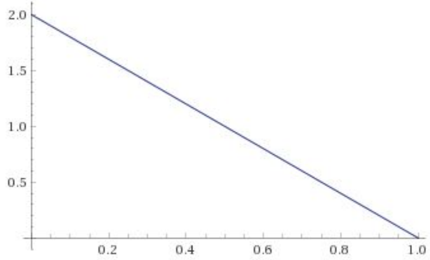
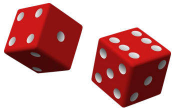
Arm (i)	True p_i	α_i	β_i	$s_{i,t}$
1	0.5	1	1	0.52
2	0.2	1	2	0.05
3	0.9	1	1	

Time (t)	Arm Pulled (a_t)	Reward (r_t)
2		

Sample from Beta(1,2) density →



Thompson Example



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1	0.5	1	1	0.52
2	0.2	1	2	0.05
3	0.9	1	1	0.67

Time (t)	Arm Pulled (a_t)	Reward (r_t)
2		

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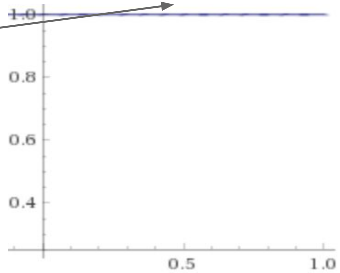

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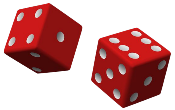

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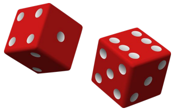
Time (t)	Arm Pulled (a_t)	Reward (r_t)
2	3	

Choose arm with highest sample!

Thompson Example



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Time (t)	Arm Pulled (a_t)	Reward (r_t)
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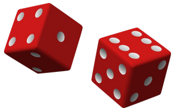
Observe reward 1 with probability 0.9 and 0 with probability 0.1.

Thompson Example



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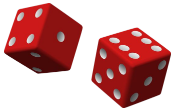
2	3	1

Add a count of 1 to the successes :).

Thompson Example



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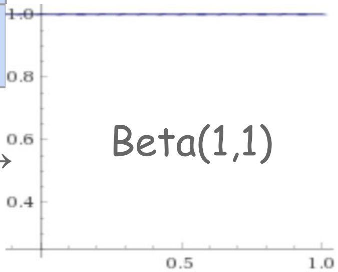
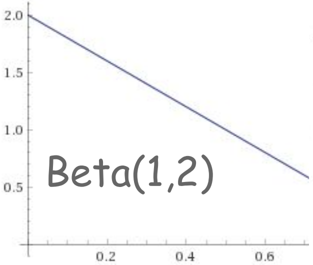
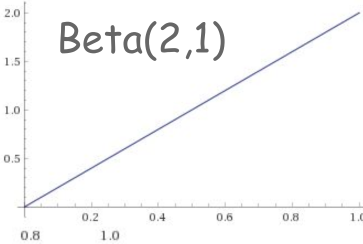
Arm (i)	True p_i	α_i	β_i	$s_{i,t}$
1	0.5	1	1	0.44
2	0.2	1	2	0.27
3	0.9	2	1	0.86

Time (t)	Arm Pulled (a_t)	Reward (r_t)
3		

Sample from each arm's Beta distribution →







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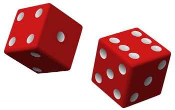

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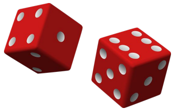

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Add a count of 1 to the failures :(.

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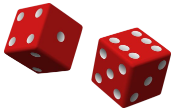

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Arm (i)	True p_i	α_i	β_i	$s_{i,t}$
1	0.5	1	1	0.63
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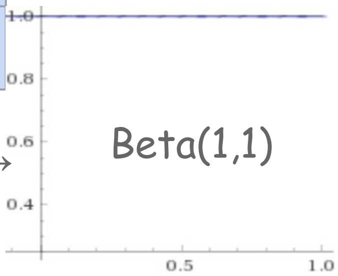
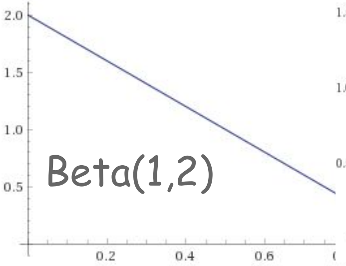
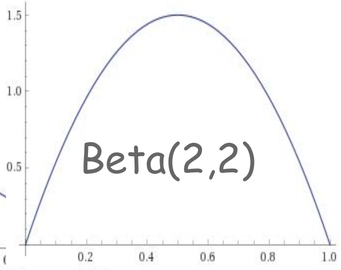
4		

Sample from each arm's Beta distribution →







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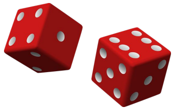

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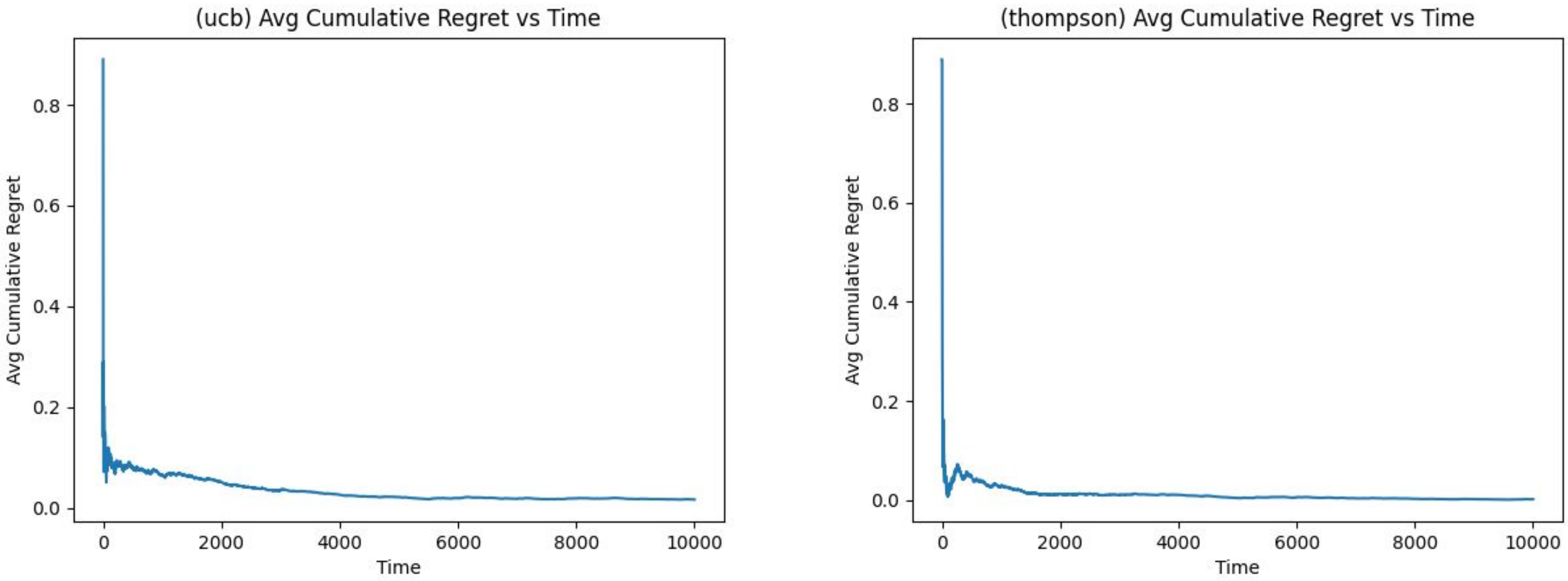
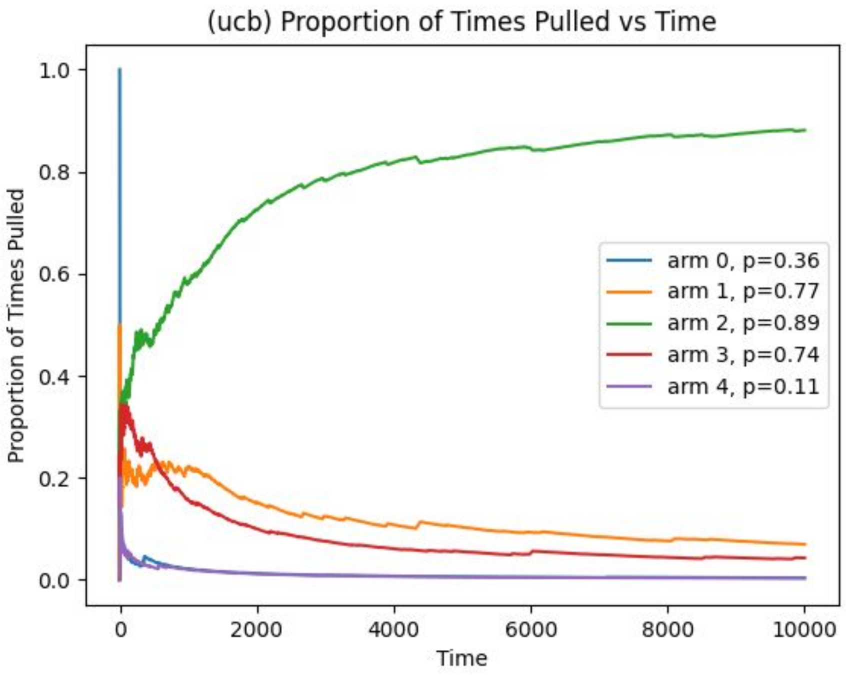
4		

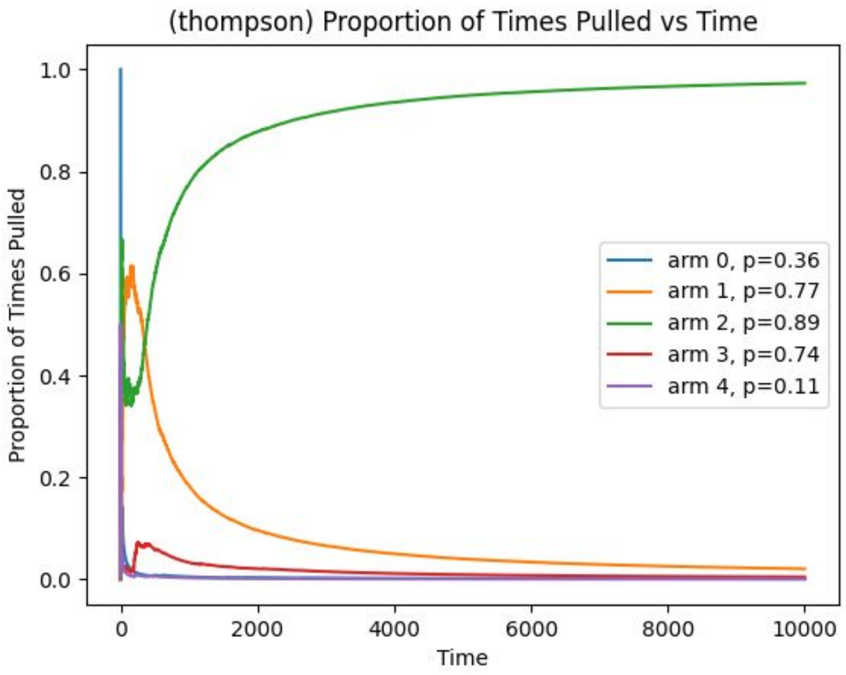
And so on!!! Notice how we explore because there's some chance the "best" arm will have a lower sample occasionally and let other arms win!

UCB Vs Thompson Sampling: Avg Regret over Time



UCB Vs Thompson Sampling: Proportion of Times Pulled





Traditional A/B (Hypothesis) Testing

A large company wants to experiment releasing a new feature/modification.

Assign
- 99% of population to control group (current feature)
- 1% to experimental group (new feature).

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Can we do better? Can we adaptively assign subjects to each group based on how each is performing rather than deciding at the beginning?

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This has the fo

- If the new … company protects i
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When feature is requested by some user, use Multi-Armed Bandit algorithm to decide which feature to show!

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(Can have any number of features/arms)

Modern A/B (Hypothesis) Testing

When to use Traditional A/B Testing:

- Need to collect data for critical business decisions.
- Need statistical confidence in all your results and impact. Want to learn even about treatments that didn't perform well.
- The reward is not immediate (e.g., if drug testing, don't have time to wait for each patient to finish before experimenting with next patient).
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The study of Multi-Armed Bandits can be categorized as:

- Statistics
- Optimization
- "Reinforcement Learning" (subfield of Machine Learning)