Maximum a-Posteriori Estimation

Aleks Jovcic
Slides by Joshua Fan & Alex Tsun

Agenda

- The Beta Random Variable
- MAP Estimation
- MAP Example

Beta Random Variable (Intuition)

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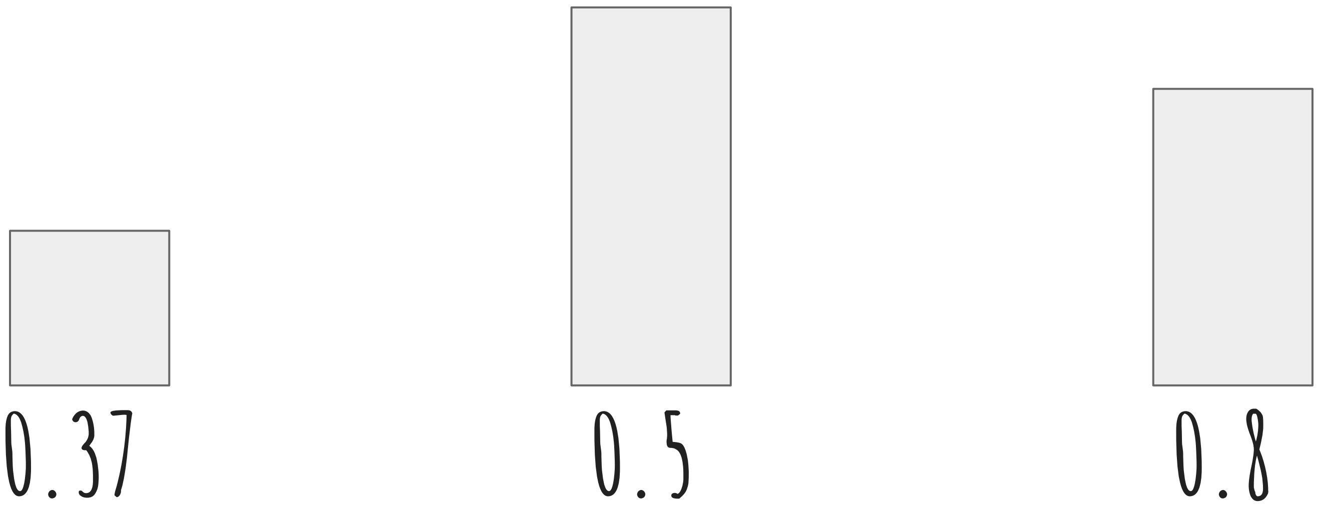

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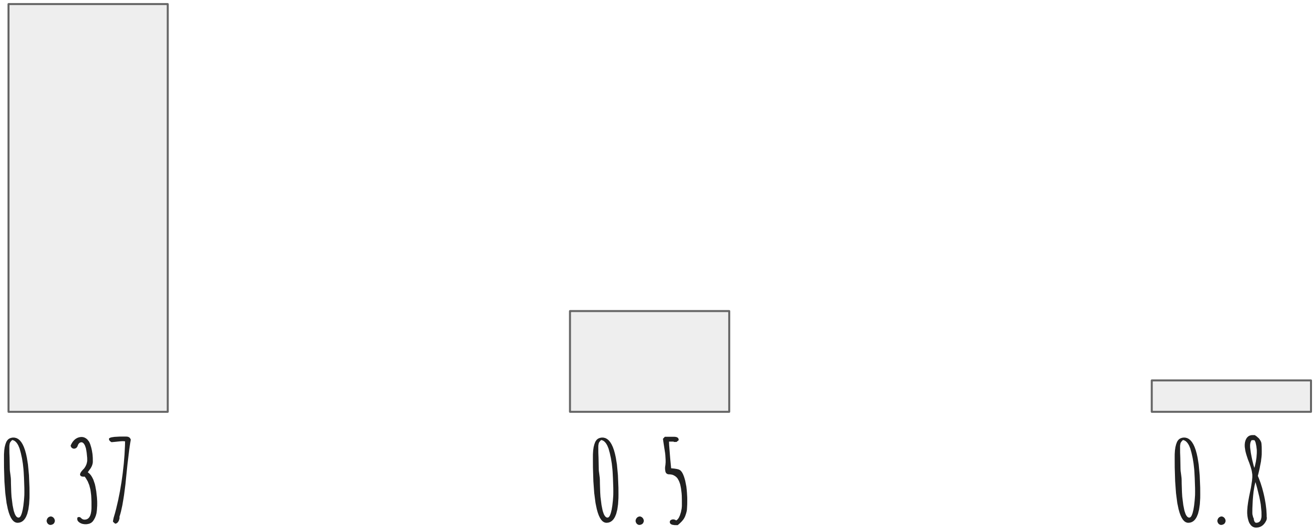

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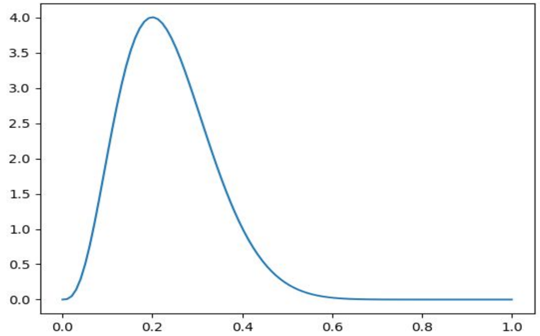

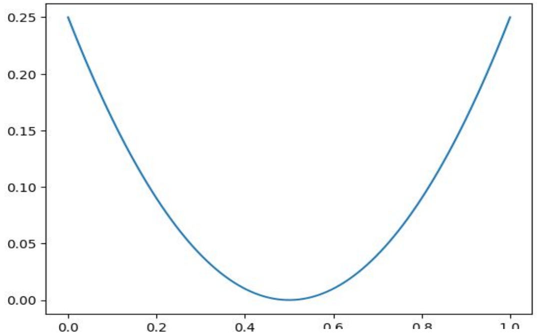

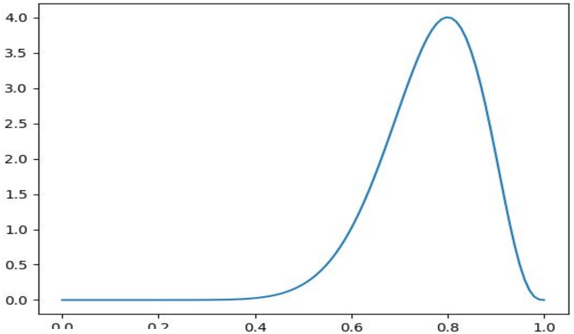

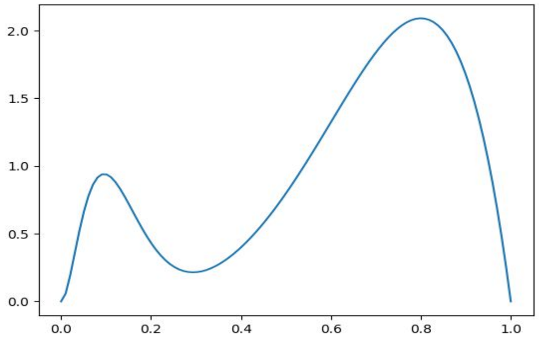

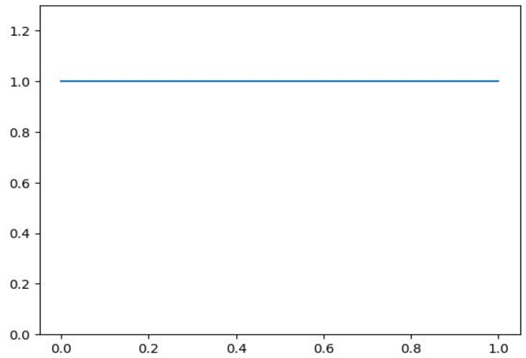

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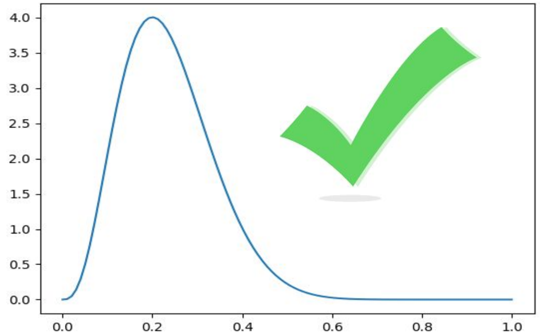

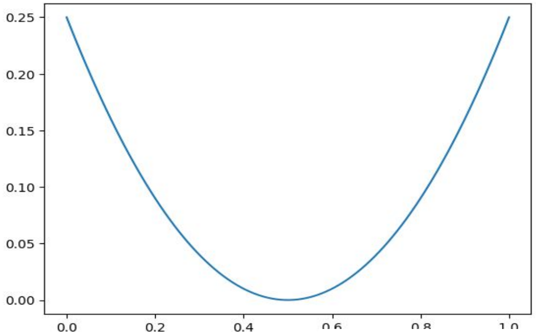

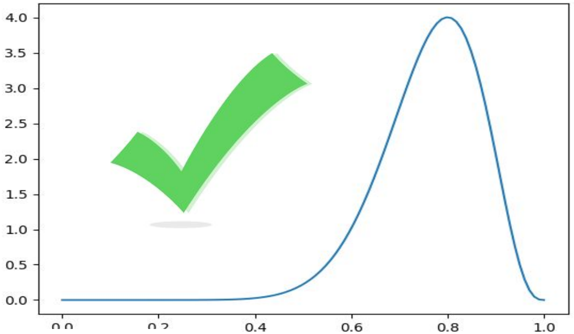

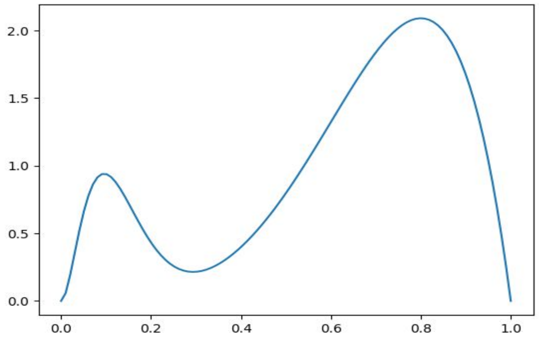

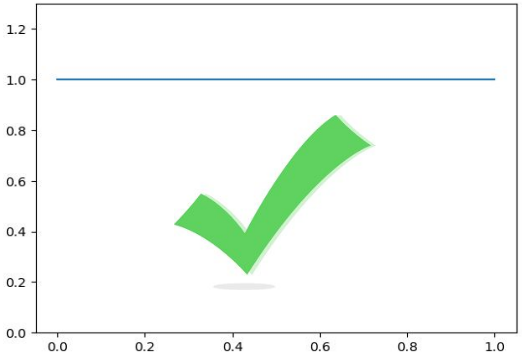

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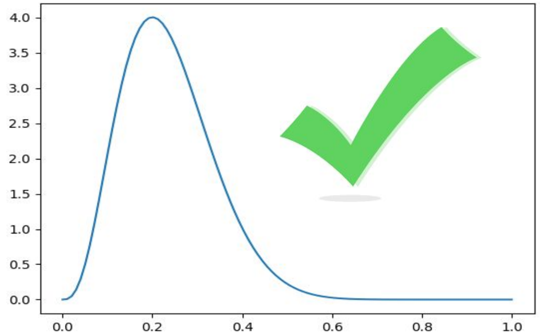

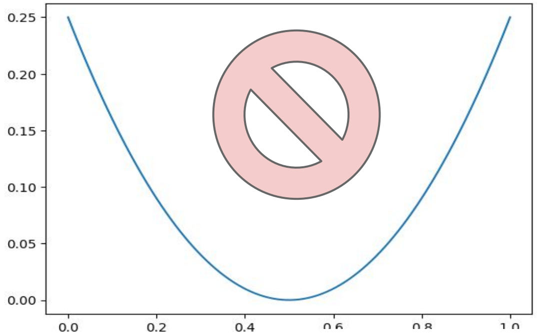

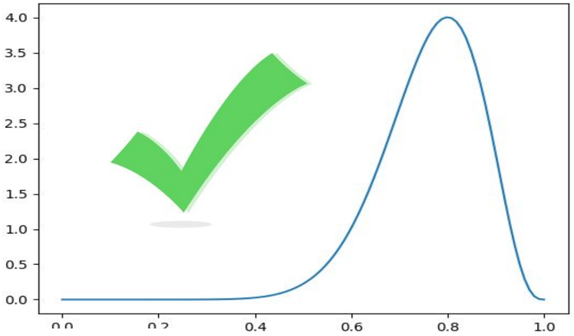

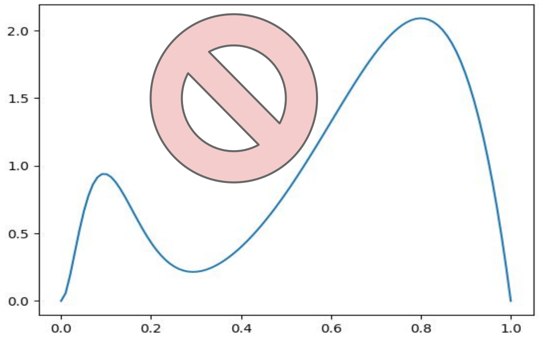

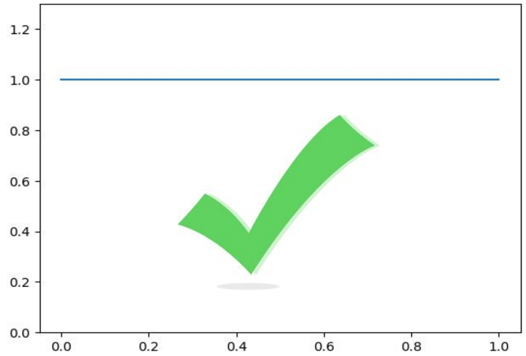

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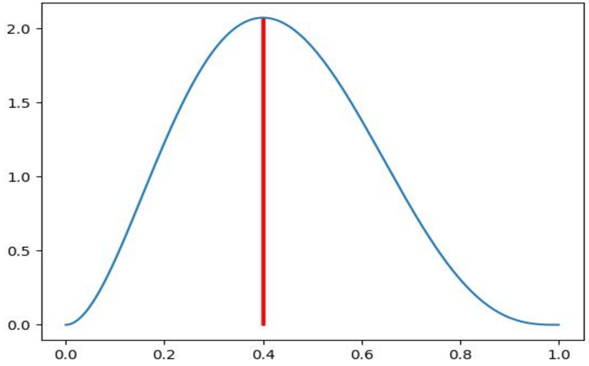

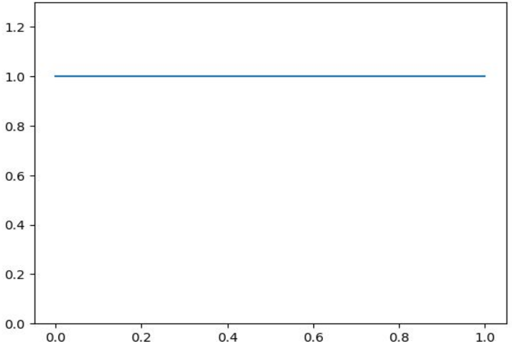

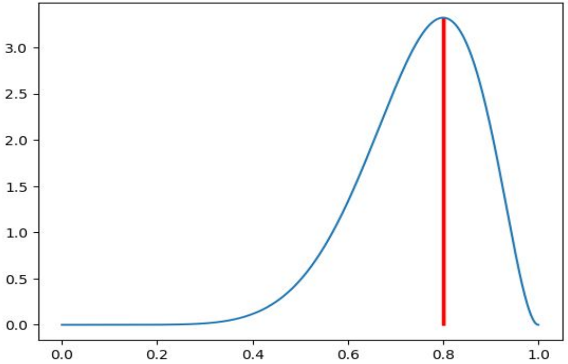

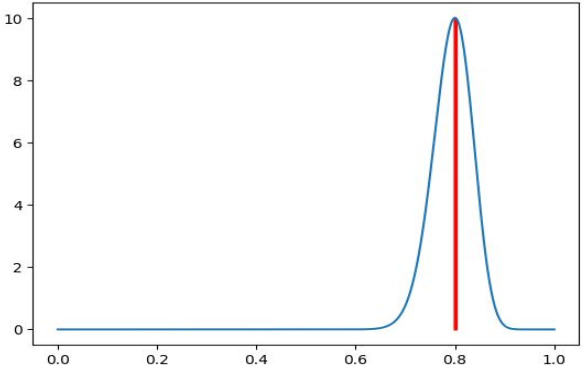

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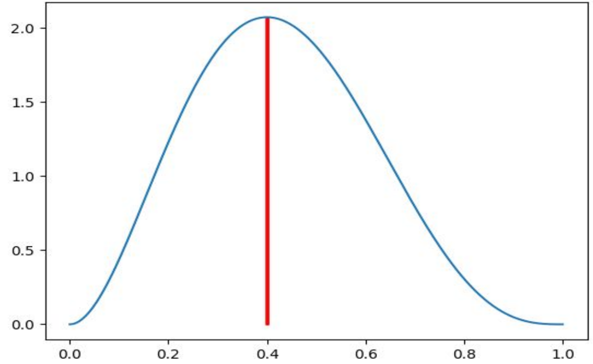

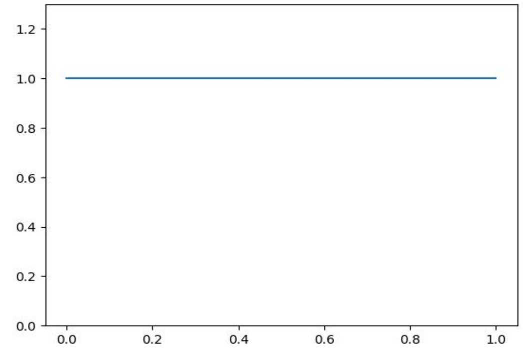

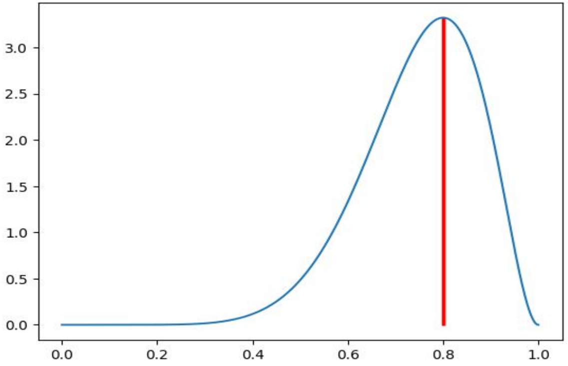

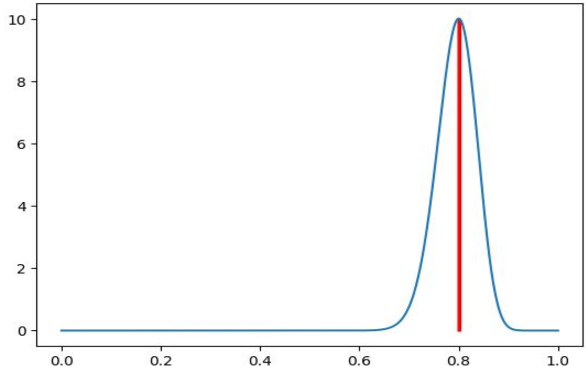

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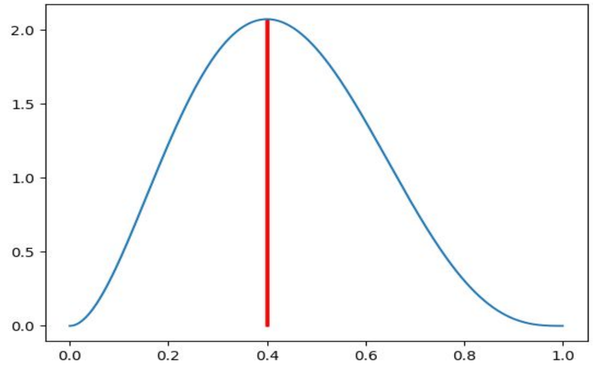

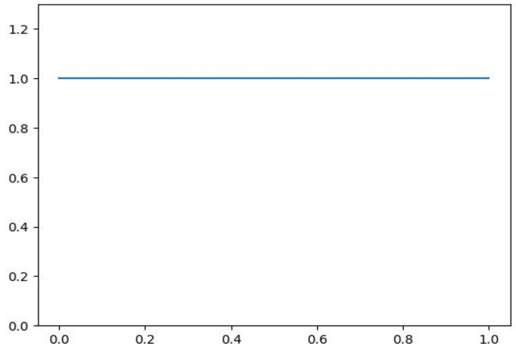

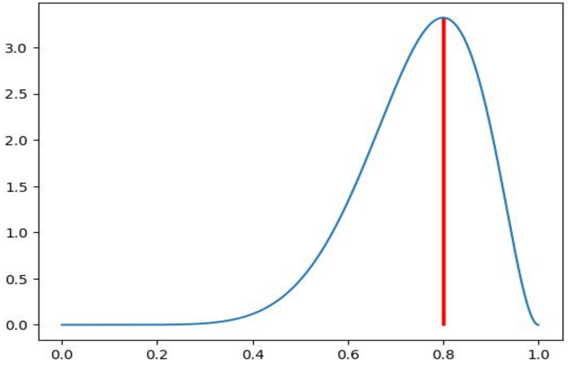

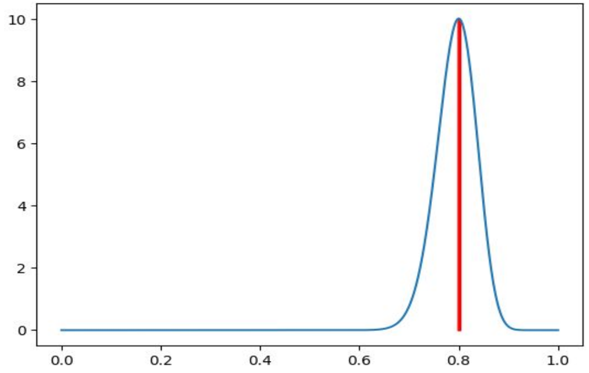

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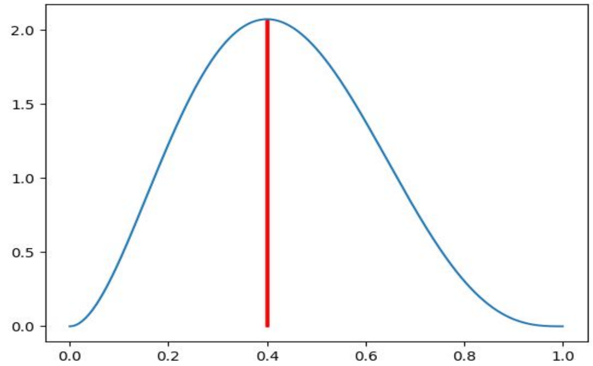

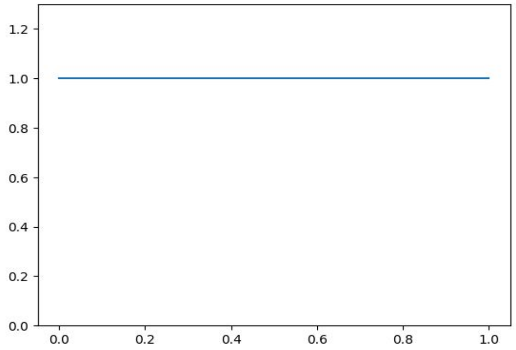

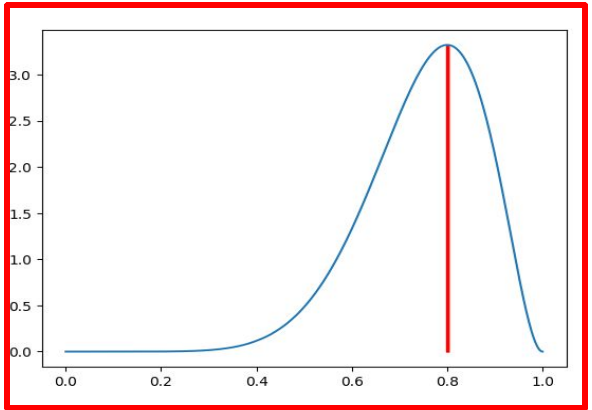

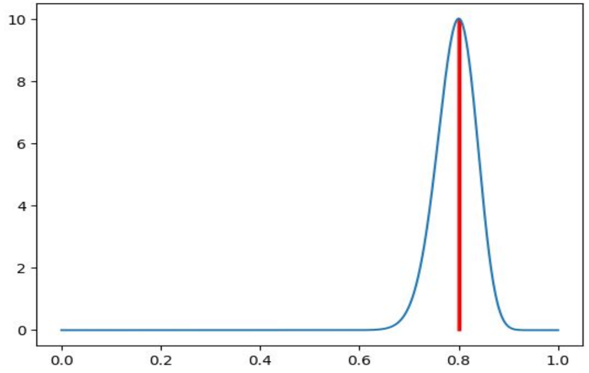

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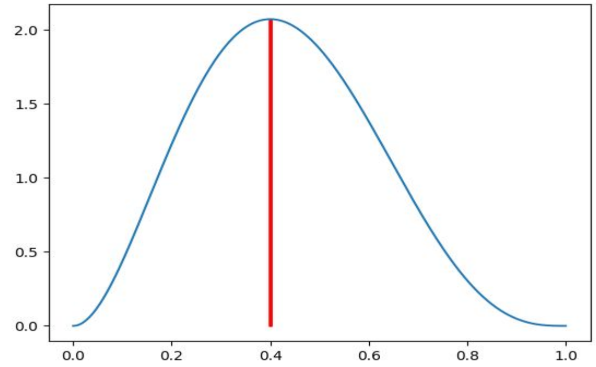

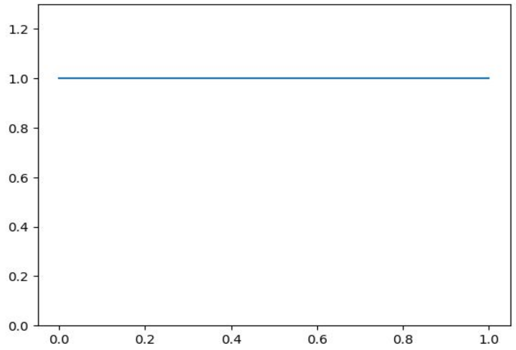

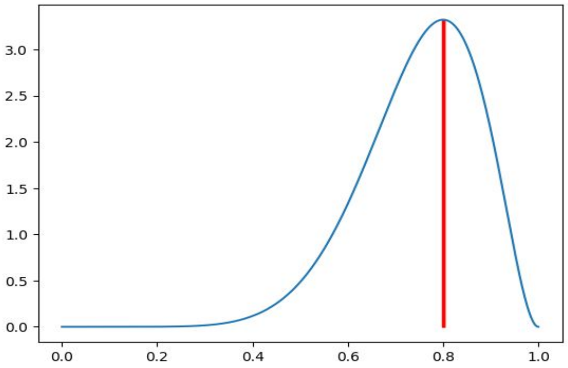

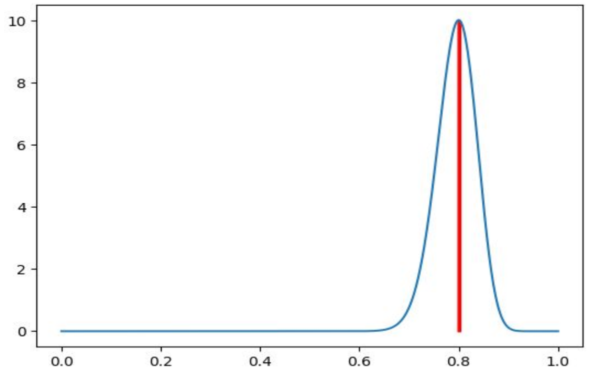

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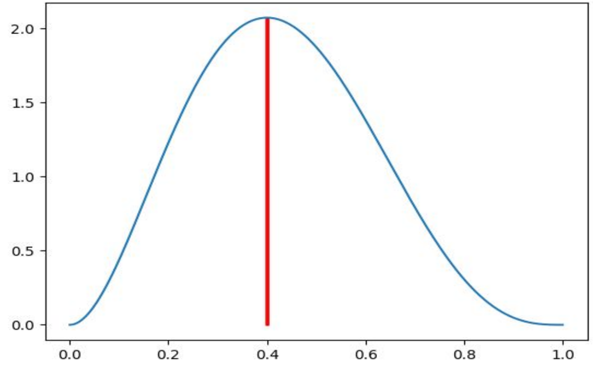

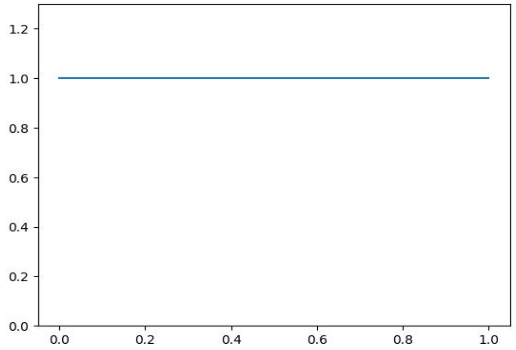

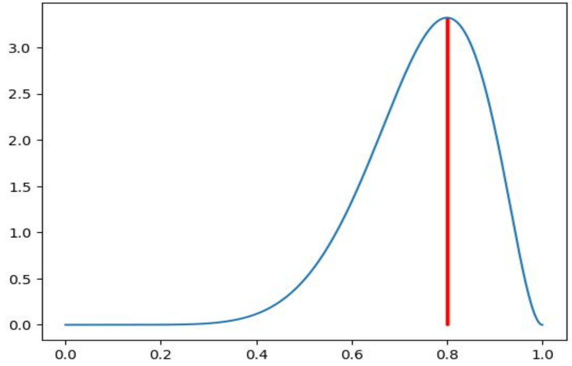

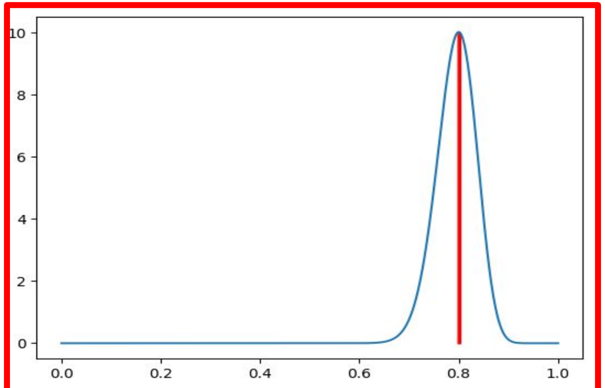

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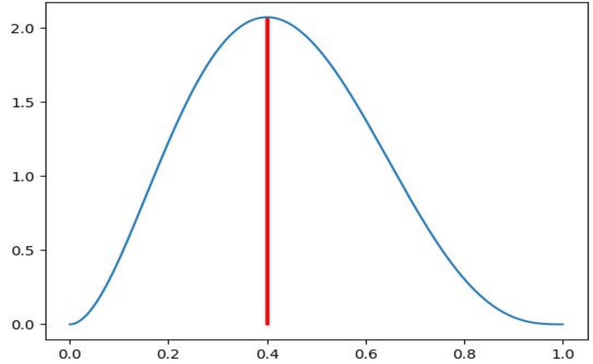

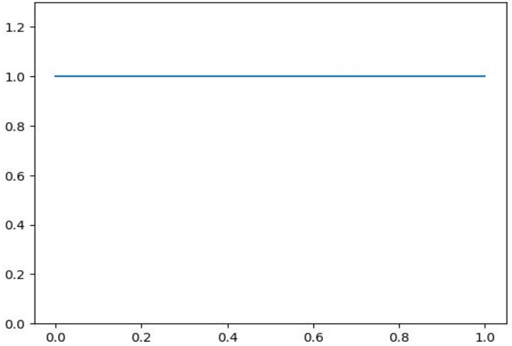

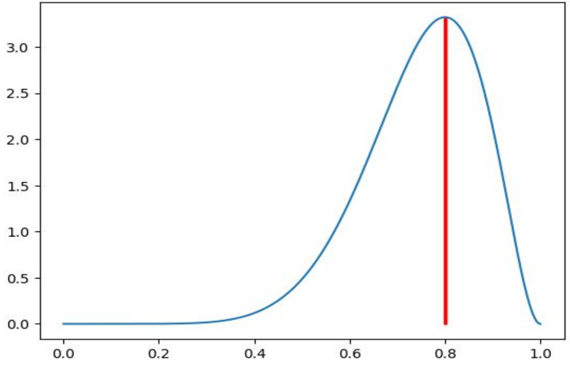

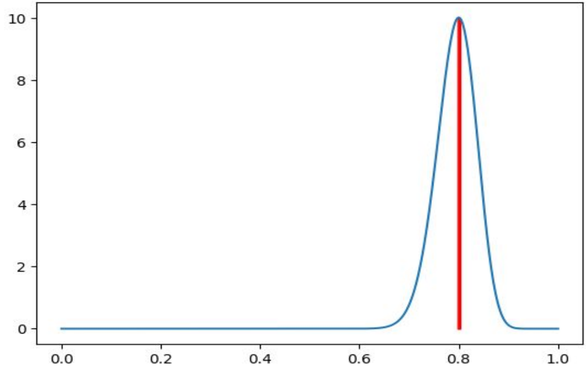

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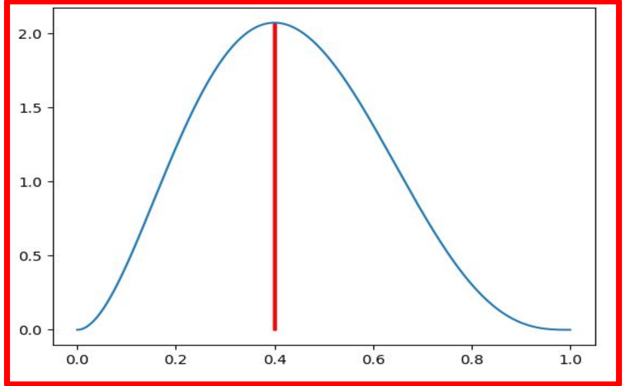

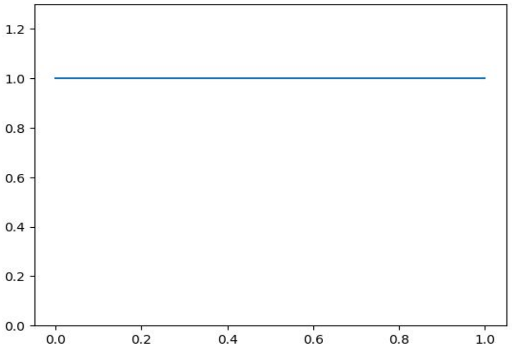

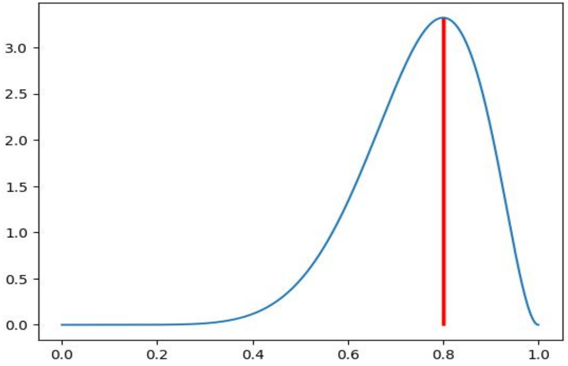

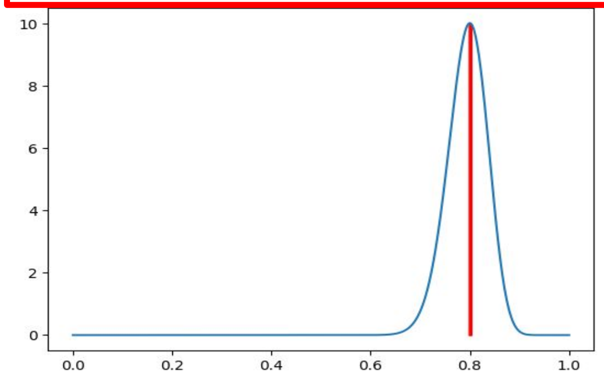

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Beta RV: $X \sim Beta(\alpha, \beta)$, if and only if X has the following pdf:

$$f_X(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

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X is typically the belief distribution about some unknown probability of success, where we pretend we saw $\alpha - 1$ successes and $\beta - 1$ failures ahead of time. Hence, the mode, $\arg\max_{x \in [0,1]} f_X(x)$, is

$$\text{mode}[X] = \frac{\alpha - 1}{(\alpha - 1) + (\beta - 1)}$$

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$$Beta(2+1, 3+1) \equiv Beta(3,4) \rightarrow \text{mode} = \frac{(3-1)}{(3-1)+(4-1)} = \frac{2}{5}$$

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Maximum a Posteriori (MAP) Estimation Idea: Actually, unknown parameter(s) is a random variable Θ. We have a ***prior*** distribution $\pi_\Theta(\theta)$ and ***posterior*** distribution (given data) $\pi_\Theta(\theta \mid \boldsymbol{x})$.

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Maximum a Posteriori (MAP) Estimation Idea: Actually, unknown parameter(s) is a random variable Θ. We have a ***prior*** distribution $\pi_\Theta(\theta)$ and ***posterior*** distribution (given data) $\pi_\Theta(\theta \mid \boldsymbol{x})$. By Bayes' Theorem,

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Maximum A Posteriori (MAP) Estimation

<u>Maximum A Posteriori (MAP) Estimation:</u> Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be iid realizations from probability mass function $p_X(t\,; \Theta = \theta)$ (if X discrete), or from density $f_X(t\,; \Theta = \theta)$ (if X continuous), where Θ is the random variable representing the parameter (or vector of parameters). We define the maximum a posteriori (MAP) estimator $\hat{\theta}_{MAP}$ of Θ to be the parameter which maximizes the posterior distribution of Θ given the data.

$$\hat{\theta}_{MAP} = \arg\max_{\theta} \pi_{\Theta}(\theta \mid \boldsymbol{x}) = \arg\max_{\theta} L(\boldsymbol{x} \mid \theta)\pi_{\Theta}(\theta)$$

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a. Suppose our samples are $x = (0,0,1,1,0)$, from $Bernoulli(\theta)$, where θ is unknown. Assume θ is unrestricted; that is, $\theta \in (0,1)$. What is the MLE for θ?

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c. Assume Θ is restricted as in part b (now a random variable for MAP). Suppose we have a (discrete) prior $\pi_\Theta(0.2) = 0.1$, $\pi_\Theta(0.5) = 0.01$, and $\pi_\Theta(0.7) = 0.89$. What is the MAP for θ?

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Choose $\pi_\Theta(\theta) = 1$ for the θ you want.

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"Lucky" Us!!

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- Anyway, in the long run, the prior ``washes out", and the only thing that matters is the likelihood; the observed data. For small sample sizes like this, the prior significantly influences the MAP estimate. However, as the number of samples goes to infinity, the MAP and MLE are equal.