CSE 312
Foundations of Computing II

Lecture 18: The Central Limit Theorem

Slide Credit: Based on Stefano Tessaro’s slides for 312 19au incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊
Agenda

• Practice with Normals
• Closure of the Normal
• The Central Limit Theorem (CLT)
Example – Off by Standard Deviations

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

\[
\mathbb{P}(|X - \mu| < k\sigma) = \frac{1}{\sqrt{2\pi}} \int_{-k\sigma}^{k\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-k\sigma}^{k\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx
\]

\[
= \phi(k) - \phi(-k)
\]

\[
= \phi(k) - (1 - \phi(k))
\]

\[
= 2\phi(k) - 1
\]
Example – Off by Standard Deviations

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$
\Pr(|X - \mu| < k\sigma) = \Pr\left(\frac{|X - \mu|}{\sigma} < k\right) = 
\Pr\left(-k < \frac{X - \mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)
$$

e.g. $k = 1: 68\%$, $k = 2: 95\%$, $k = 3: 99\%$
Summary of procedure for doing calculations with normal r.v.

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

Therefore,

$$F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P}\left(\frac{X - \mu}{\sigma} \leq \frac{z - \mu}{\sigma}\right) = \Phi\left(\frac{z - \mu}{\sigma}\right)$$
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CDF of normal distribution

Fact. If \( X \sim \mathcal{N}(\mu, \sigma^2) \), then \( Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2) \)

Standard (unit) normal \( Z \sim \mathcal{N}(0, 1) \)

CDF. \( \Phi(z) = \mathbb{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \, dx \) for \( Z \sim \mathcal{N}(0, 1) \)

Note: \( \Phi(z) \) has no closed form – generally given via tables

If \( X \sim \mathcal{N}(\mu, \sigma^2) \), then \( F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right) \)
Closure of the normal -- under addition

**Fact.** If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV) then $aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that the sum of normal RVs is still a normal RV.

The values of the expectation and variance is not surprising.
• Linearity of expectation (always true)
• When $X$ and $Y$ are independent, $Var(aX + bY) = a^2Var(X) + b^2Var(Y)$
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Gaussian in Nature

Empirical distribution of collected data often resembles a Gaussian ...

- e.g. Height distribution resembles Gaussian.

R.A. Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can written as

\[ X = X_1 + \cdots + X_n \]
Sum of Independent RVs

$X_1, \ldots, X_n$ i.i.d. with expectation $\mu$ and variance $\sigma^2$

Define

$$S_n = X_1 + \cdots + X_n$$

$$\mathbb{E}(S_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = n\mu$$

$$\text{Var}(S_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n) = n\sigma^2$$

**Empirical observation:** $S_n$ looks like a normal RV as $n$ grows.
CLT (Idea)

From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf
Sum of i.i.d. exponential random variables (param 1)
CLT (Idea)

From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf
Central Limit Theorem

$X_1, \ldots, X_n$ i.i.d., each with expectation $\mu$ and variance $\sigma^2$

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}(Y_n) = \mathbb{E}\left[\frac{S_n - n\mu}{\sigma\sqrt{n}}\right] = \frac{1}{\sigma\sqrt{n}} \mathbb{E}[S_n] - n\mu = 0$$

$$\text{Var}(Y_n) = \text{Var}\left(\frac{S_n - n\mu}{\sigma\sqrt{n}}\right) = \frac{1}{\sigma^2\sqrt{n}} \text{Var}(S_n) = \frac{\sigma^2}{\sigma^2 \sqrt{n}} = 1$$
Central Limit Theorem

\(X_1, \ldots, X_n\) i.i.d., each with expectation \(\mu\) and variance \(\sigma^2\)

Define \(S_n = X_1 + \cdots + X_n\) and

\[
Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}
\]

\[
\mathbb{E}(Y_n) = \frac{1}{\sigma\sqrt{n}} (\mathbb{E}(S_n) - n\mu) = \frac{1}{\sigma\sqrt{n}} (n\mu - n\mu) = 0
\]

\[
\text{Var}(Y_n) = \frac{1}{\sigma^2n} (\text{Var}(S_n - n\mu)) = \frac{\text{Var}(S_n)}{\sigma^2n} = \frac{\sigma^2n}{\sigma^2n} = 1
\]
Central Limit Theorem

**Theorem. (Central Limit Theorem)** The CDF of $Y_n$ converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$
\lim_{n \to \infty} \mathbb{P}(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \, dx
$$

$$
Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}}
$$
Central Limit Theorem

\[ Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}} \]

\[ \overline{F_X}(x) = \mathbb{P}(X < x) \]

**Theorem. (Central Limit Theorem)** The CDF of \( Y_n \) converges to the CDF of the standard normal \( \mathcal{N}(0,1) \), i.e.,

\[
\lim_{n \to \infty} \mathbb{P}(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \, dx
\]

Also stated as:

- \( \lim_{n \to \infty} Y_n \to \mathcal{N}(0,1) \)
- \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i \to \mathcal{N} \left( \mu, \frac{\sigma^2}{n} \right) \) where \( \mu = \mathbb{E}[X_i] \) and \( \sigma^2 = \text{Var}(X_i) \)
CLT → Normal Distribution EVERYWHERE

Neuron Activity

S&P 500 Returns after Elections

Vegetables

Examples from:
https://galtonboard.com/probabilityexamplesinlife
Example

We flip \( n \) independent coins, heads with probability \( p = 0.75 \).

\( X = \# \text{ heads} \quad \mu = \mathbb{E}(X) = 0.75n \quad \sigma^2 = \text{Var}(X) = 0.1875n \)

\[ n = 0 \quad -7.5 \quad 1.875 \]

\[ \mathbb{P}(X \leq 7) = \mathbb{P}(0 \leq X \leq 7) \]

\[ \mathbb{P}(X \leq 0.7n) \]

= \( \Phi \left( \frac{-7.5}{1.875} \right) - \Phi \left( \frac{-0.5}{\sqrt{0.1875}} \right) = 0 - \Phi \)
Example

We flip $n$ independent coins, heads with probability $p = 0.75$.

$X = \# \text{heads} \quad \mu = \mathbb{E}(X) = 0.75n \quad \sigma^2 = \text{Var}(X) = 0.1875n$

$\mathbb{P}(X \leq 0.7n)$

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