CSE 312
Foundations of Computing II

Lecture 18: The Central Limit Theorem

Slide Credit: Based on Stefano Tessaro’s slides for 312 19au incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊
Agenda

• Practice with Normals
• Closure of the Normal
• The Central Limit Theorem (CLT)
Example – Off by Standard Deviations

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$
\mathbb{P}(|X - \mu| < k\sigma) =
$$
Example – Off by Standard Deviations

Let $X \sim \mathcal{N}(\mu, \sigma^2)$.

$$\mathbb{P}(|X - \mu| < k\sigma) = \mathbb{P}\left(\frac{|X - \mu|}{\sigma} < k\right) =$$

$$= \mathbb{P}\left(-k < \frac{X - \mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k)$$

e.g. $k = 1$: 68%, $k = 2$: 95%, $k = 3$: 99%
Summary of procedure for doing calculations with normal r.v.

If $X \sim \mathcal{N} \left( \mu, \sigma^2 \right)$, then $\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

Therefore,

$$F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P} \left( \frac{X - \mu}{\sigma} \leq \frac{z - \mu}{\sigma} \right) = \Phi \left( \frac{z - \mu}{\sigma} \right)$$
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CDF of normal distribution

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

Standard (unit) normal $Z \sim \mathcal{N}(0, 1)$

CDF. $\Phi(z) = \mathbb{P}(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} \, dx$ for $Z \sim \mathcal{N}(0, 1)$

Note: $\Phi(z)$ has no closed form – generally given via tables

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $F_X(z) = \mathbb{P}(X \leq z) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$
Closure of the normal -- under addition

**Fact.** If $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$, $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ (both independent normal RV) then $aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that the sum of normal RVs is still a normal RV.

The values of the expectation and variance is not surprising.
• Linearity of expectation (always true)
• When $X$ and $Y$ are independent, $\text{Var}(aX + bY) = a^2 \text{Var}(X) + b^2 \text{Var}(Y)$
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Gaussian in Nature

Empirical distribution of collected data often resembles a Gaussian ...

e.g. Height distribution resembles Gaussian.

R.A.Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can written as

\[ X = X_1 + \cdots + X_n \]
Sum of Independent RVs

\[ X_1, \ldots, X_n \text{ i.i.d. with expectation } \mu \text{ and variance } \sigma^2 \]

Define

\[ S_n = X_1 + \cdots + X_n \]

\[ \mathbb{E}(S_n) = \mathbb{E}(X_1) + \cdots + \mathbb{E}(X_n) = n\mu \]

\[ \text{Var}(S_n) = \text{Var}(X_1) + \cdots + \text{Var}(X_n) = n\sigma^2 \]

**Empirical observation:** \( S_n \) looks like a normal RV as \( n \) grows.
CLT (Idea)

From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf
Sum of i.i.d. exponential random variables (param 1)
CLT (Idea)

From: https://courses.cs.washington.edu/courses/cse312/17wi/slides/10limits.pdf
Central Limit Theorem

$X_1, \ldots, X_n$ i.i.d., each with expectation $\mu$ and variance $\sigma^2$

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$\mathbb{E}(Y_n) =$

$\text{Var}(Y_n) =$
Central Limit Theorem

$X_1, \ldots, X_n$ i.i.d., each with expectation $\mu$ and variance $\sigma^2$

Define $S_n = X_1 + \cdots + X_n$ and

$$Y_n = \frac{S_n - n\mu}{\sigma \sqrt{n}}$$

$$\mathbb{E}(Y_n) = \frac{1}{\sigma \sqrt{n}} (\mathbb{E}(S_n) - n\mu) = \frac{1}{\sigma \sqrt{n}} (n\mu - n\mu) = 0$$

$$\text{Var}(Y_n) = \frac{1}{\sigma^2 n} (\text{Var}(S_n - n\mu)) = \frac{\text{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$
Theorem. (Central Limit Theorem) The CDF of $Y_n$ converges to the CDF of the standard normal $\mathcal{N}(0,1)$, i.e.,

$$
\lim_{n \to \infty} \mathbb{P}(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \, dx
$$

where

$$
Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}}
$$
Central Limit Theorem

\[ Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} \]

**Theorem. (Central Limit Theorem)** The CDF of \( Y_n \) converges to the CDF of the standard normal \( \mathcal{N}(0,1) \), i.e.,

\[
\lim_{n \to \infty} \mathbb{P}(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-x^2/2} \, dx
\]

Also stated as:

- \( \lim_{n \to \infty} Y_n \to \mathcal{N}(0,1) \)
- \( \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i \to \mathcal{N} \left( \mu, \frac{\sigma^2}{n} \right) \) where \( \mu = E[X_i] \) and \( \sigma^2 = Var(X_i) \)
CLT $\rightarrow$ Normal Distribution EVERYWHERE

Neuron Activity

S&P 500 Returns after Elections

Vegetables

Examples from: https://galtonboard.com/probabilityexamplesinlife
Example

We flip $n$ independent coins, heads with probability $p = 0.75$.

$X = \# \text{ heads} \quad \mu = \mathbb{E}(X) = 0.75n \quad \sigma^2 = \text{Var}(X) = 0.1875n$

$P(X \leq 0.7n)$
Example

We flip $n$ independent coins, heads with probability $p = 0.75$.

$X = \# \text{heads} \quad \mu = \mathbb{E}(X) = 0.75n \quad \sigma^2 = \text{Var}(X) = 0.1875n$

\[ P(X \leq 0.7n) \]

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