CSE 312

# Foundations of Computing II

## Lecture 16: Joint Continuous, Conditional Distributions, and Tail Bounds

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Anna Karlin, Ryan O'Donnell, Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

# Agenda

- Joint Continuous Distributions
- Conditional Expectation
  - Law of Total Expectation
- Tail Bounds
  - Markov's Inequality
  - Chebyshev's Inequality

| | Discrete | Continuous |
|---|---|---|
| **Joint PMF/PDF** | $p_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y)$ | $f_{X,Y}(x,y) \neq \mathbb{P}(X=x, Y=y)$ |
| **Joint range/support** $\Omega_{X,Y}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$ | $\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$ |
| **Joint CDF** | $F_{X,Y}(x,y) = \sum_{t \le x, s \le y} p_{X,Y}(t,s)$ | $F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$ |
| **Normalization** | $\sum_{x,y} p_{X,Y}(x,y) = 1$ | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$ |
| **Marginal PMF/PDF** | $p_X(x) = \sum_y p_{X,Y}(x,y)$ | $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$ |
| **Expectation** | $\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$ | $\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$ |

# Independence (continuous random variables)

**Definition.** Let $X$ and $Y$ be continuous random variables. The **joint pdf** of $X$ and $Y$ is

$$f_{X,Y}(a,b) \neq \Pr(X=a, Y=b)$$

**Definition.** The **joint range** of $p_{X,Y}$ is

$$\Omega(X,Y) = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega(X) \times \Omega(Y)$$

**Definition.** $X$ and $Y$ are **independent** iff for all $a, b$

$$f_{X,Y}(a,b) = f_X(a) \cdot f_Y(b)$$

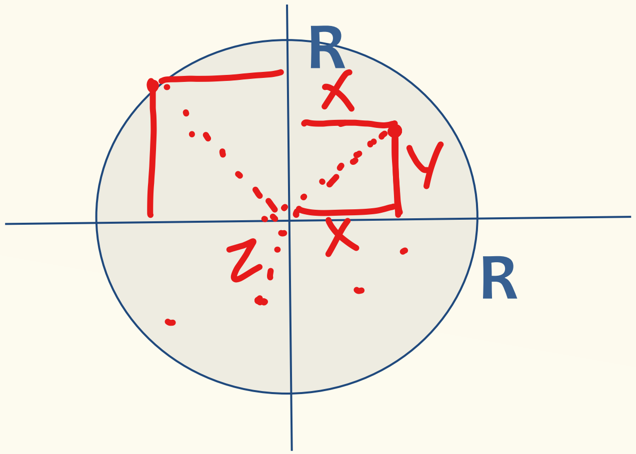
- Suppose that the surface of a disk is a circle with ~~area~~ radius R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is their joint density f(x,y)?

$$f_{X,Y} = c \quad x^2 + y^2 \le R^2$$

$$f_{X,Y}\left(\frac{R}{2}, \frac{R}{3}\right) = \frac{1}{\pi R^2}$$

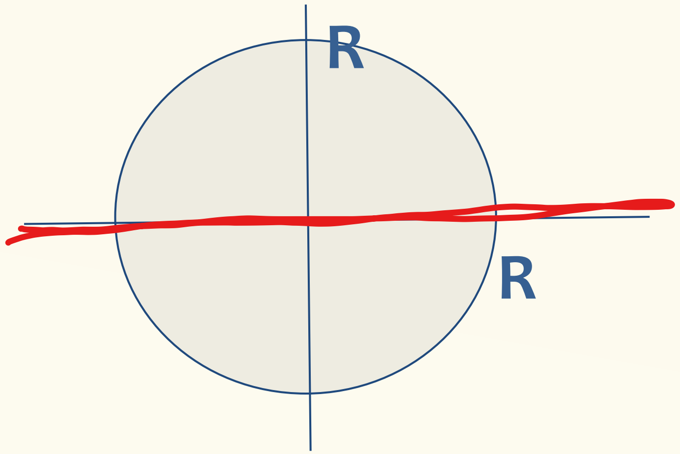
$$\iint_{\pi R^2} c = 1 = c\pi R^2 = 1$$

$$c = \frac{1}{\pi R^2}$$



- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is the range of X & Y and the marginal density of X and of Y?

$\Omega_X : [- , R]$




Poll:

What is $\Omega_X$?

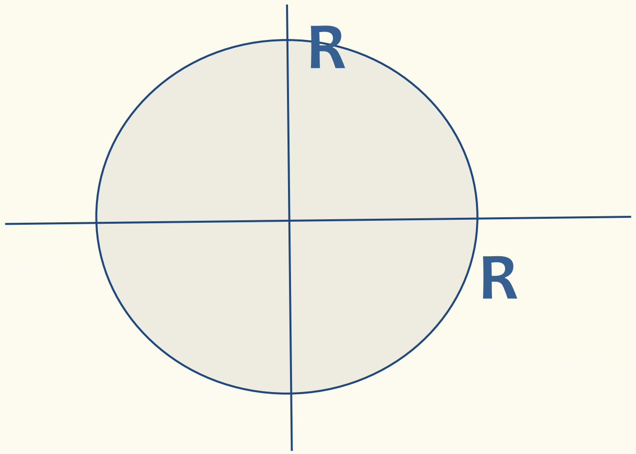
a. $[-\sqrt{R^2 - x^2}, \sqrt{R^2 - x^2}]$

b. $[-R, R]$

c. $[-\sqrt{R^2 - y^2}, \sqrt{R^2 - y^2}]$

d. Not sure

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - Are X and Y independent?

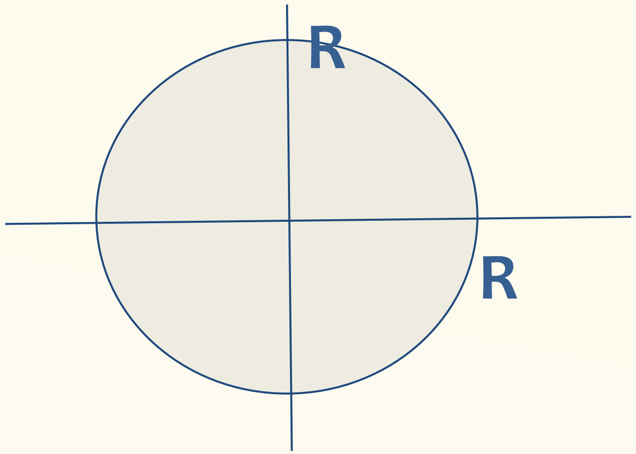



Poll:

Are X and Y independent?

a. yes
b. no

- Suppose that the surface of a disk is a circle with area R centered at the origin and that there is a single point imperfection at a location with is uniformly distributed across the surface of the disk. Let X and Y be the x and y coordinates of the imperfection (random variables) and let Z be the distance of the imperfection from the origin.
  - What is E(Z)?



# All of this generalizes to more than 2 random variables

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| **Normalization** | $\sum_{x,y} p_{X,Y}(x,y) = 1$ | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$ |
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  - Law of Total Expectation
- Tail Bounds
  - Markov's Inequality
  - Chebyshev's Inequality

# Conditional Expectation

**Definition.** Let $X$ be a discrete random variable then the **conditional expectation** of $X$ given event $A$ is

$$E[X \mid A] = \sum_{x \in \Omega(X)} x \Pr(X = x \mid A)$$

- Linearity of expectation still applies here

$$E[aX + bY + c \mid A] = aE[X \mid A] + bE[Y \mid A] + c$$

# Conditional Expectation

**Definition.** Let $X$ be a discrete random variable then the **conditional expectation** of $X$ given event $Y = y$ is

$$E[X \mid Y = y] = \sum_{x \in \Omega(X)} x \Pr(X = x \mid Y = y)$$

- Linearity of expectation still applies here

$$E[aX + bY + c \mid Y = y] = aE[X \mid Y = y] + bE[Y \mid Y = y] + c$$

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# Law of Total Expectation

**Law of Total Expectation (event version).** Let $X$ be a random variable and let events $A_1, \dots, A_n$ partition the sample space. Then,

$$E[X] = \sum_{i=1}^{n} E[X|A_i]\Pr(A_i)$$

# Proof of Law of Total Expectation

Follows from Law of Total Probability and manipulating sums

$$E[X] = \sum_{x \in \Omega(X)} x \Pr(X = x)$$

$$= \sum_{x \in \Omega(X)} x \sum_{i=1}^{n} \Pr(X = x \mid A_i) \Pr(A_i)$$ (by LTP)

$$= \sum_{i=1}^{n} \Pr(A_i) \sum_{x \in \Omega(X)} x \Pr(X = x \mid A_i)]$$ (change order of sums)

$$= \sum_{i=1}^{n} \Pr(A_i) E[X \mid A_i]$$ (def of cond. expect.)

# Law of Total Expectation

**Law of Total Expectation (random variable version).** Let $X$ be a random variable and $Y$ be a discrete random variable. Then,

$$E[X] = \sum_{y \in \Omega(Y)} E[X|Y = y]\Pr(Y = y)$$

# Example: Flipping Coins

Suppose wanted to analyze flipping a random number of coins. Suppose someone gave us $Y \sim Poi(5)$ fair coins and we wanted to compute the expected number of heads $X$ from flipping those coins.

# Elevator rides

The number of people who enter an elevator on the ground floor is a Poisson random variable with mean 10. If there are N floors above the ground floor, and if each person is equally likely to get off at any one of the N floors, independently of where others get off, compute the expected number of stops the elevator will make before discharging all the passengers.

# Reference Sheet (with continuous RVs)

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| Conditional PMF/PDF | $p_{X \mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$ | $f_{X \mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$ |
| Conditional Expectation | $E[X \mid Y=y] = \sum_x x p_{X \mid Y}(x \mid y)$ | $E[X \mid Y=y] = \int_{-\infty}^{\infty} x f_{X \mid Y}(x \mid y) dx$ |
| Independence | $\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ | $\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ |

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# Tail Bounds (Idea)

Bounding the probability a random variable is far from its mean. Usually statements of the form:

$$\Pr(X \geq a) \leq b$$
$$\Pr(|X - E[X]| \geq a) \leq b$$

Useful tool when

- An approximation that is easy to compute is sufficient
- The process is too complex to analyze exactly

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# Markov's Inequality

**Theorem.** Let $X$ be a random variable taking only non-negative values. Then, for any $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

$$\mathbb{P}\big(X \geq t \cdot \mathbb{E}(X)\big) \leq \frac{1}{t}.$$

Incredibly simplistic – only requires that the random variable is non-negative and only needs you to know expectation. You don't need to know **anything else** about the distribution of $X$.

# Markov's Inequality – Proof

**Theorem.** Let $X$ be a (discrete) random variable taking only non-negative values. Then, for any $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

$$\mathbb{E}(X) = \sum_{x} x \cdot \mathbb{P}(X = x)$$

$$= \sum_{x \geq t} x \cdot \mathbb{P}(X = x) + \sum_{x < t} x \cdot \mathbb{P}(X = x)$$

# Markov's Inequality – Proof

**Theorem.** Let $X$ be a (discrete) random variable taking only non-negative values. Then, for any $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

$$\mathbb{E}(X) = \sum_{x} x \cdot \mathbb{P}(X = x)$$

$$= \sum_{x \geq t} x \cdot \mathbb{P}(X = x) + \sum_{x < t} x \cdot \mathbb{P}(X = x)$$

$\geq 0$ because $x \geq 0$ whenever $\mathbb{P}(X = x) \geq 0$ (takes only non-negative values)

$$\geq \sum_{x \geq t} x \cdot \mathbb{P}(X = x)$$

$$\geq \sum_{x \geq t} t \cdot \mathbb{P}(X = x) \quad = t \cdot \mathbb{P}(X \geq t)$$

Follows by re-arranging terms …

# Example – Binomial Random Variable

Markov's inequality
$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$

Let $X$ be Binomial RV with parameters. $n, p$

$$\mathbb{E}(X) = \frac{n}{2}$$

*What is the probability that* $X \geq \frac{3n}{4}$?

Markov's inequality: $\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{3n} \cdot \frac{n}{2} = \frac{2}{3}$

**Can we do better?**

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# Using variance

- If we know more about the random variable, e.g. its variance, we can get a better bound!

# Chebyshev's Inequality

Markov's inequality
$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}$.

**Theorem.** Let $X$ be a random variable. Then, for any $t > 0$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}.$$

**Proof:** Define $Z = X - \mathbb{E}(X)$

$$\mathbb{P}(|Z| \geq t) = \mathbb{P}(Z^2 \geq t^2) \leq \frac{\mathbb{E}(Z^2)}{t^2} = \frac{\mathrm{Var}(X)}{t^2}$$

Definition of Variance

$|Z| \geq t$ iff $Z^2 \geq t^2$

Markov's inequality ($Z^2 \geq 0$)

# Example – Binomial Random Variable

Chebychev's Inequality
$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}$.

Let $X$ be Binomial RV with parameters. $n, p = 0.5$

$\mathbb{E}(X) = \frac{n}{2}$ $\quad$ $Var(X) =$

*What is the probability that* $X \geq \frac{3n}{4}$*?*

Chebychev's inequality: $\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq$

Markov's inequality: $\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq \frac{4}{3n} \cdot \frac{n}{2} = \frac{2}{3}$

# Tail Bounds

Useful for approximations of complex systems. How good the approximation is depends on the actual distribution and the context you are using it in.

- Usually loose upper-bounds are okay when designing for worst-case

Generally, the more you know about your random variable the better tail bounds you can get.