CSE 312

Foundations of Computing II

Lecture 13: Continuous RVs and the Exponential Distribution

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

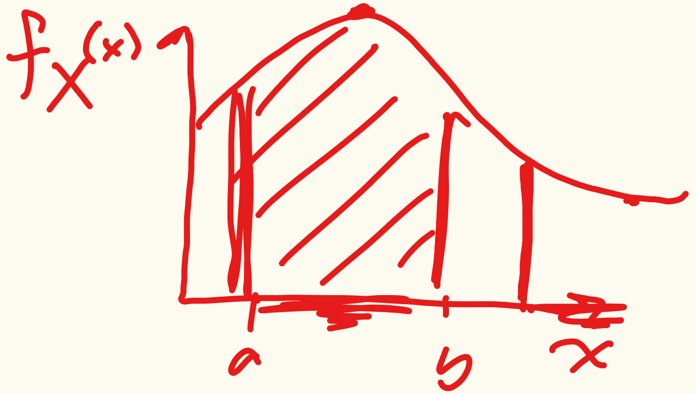
Agenda

- Continuous RVs
  - Cumulative Distribution Function
  - Expectation and Variance
- Exponential Distribution
- Time permitting: Memorylessness

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \to \mathbb{R}$, such that

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$



$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

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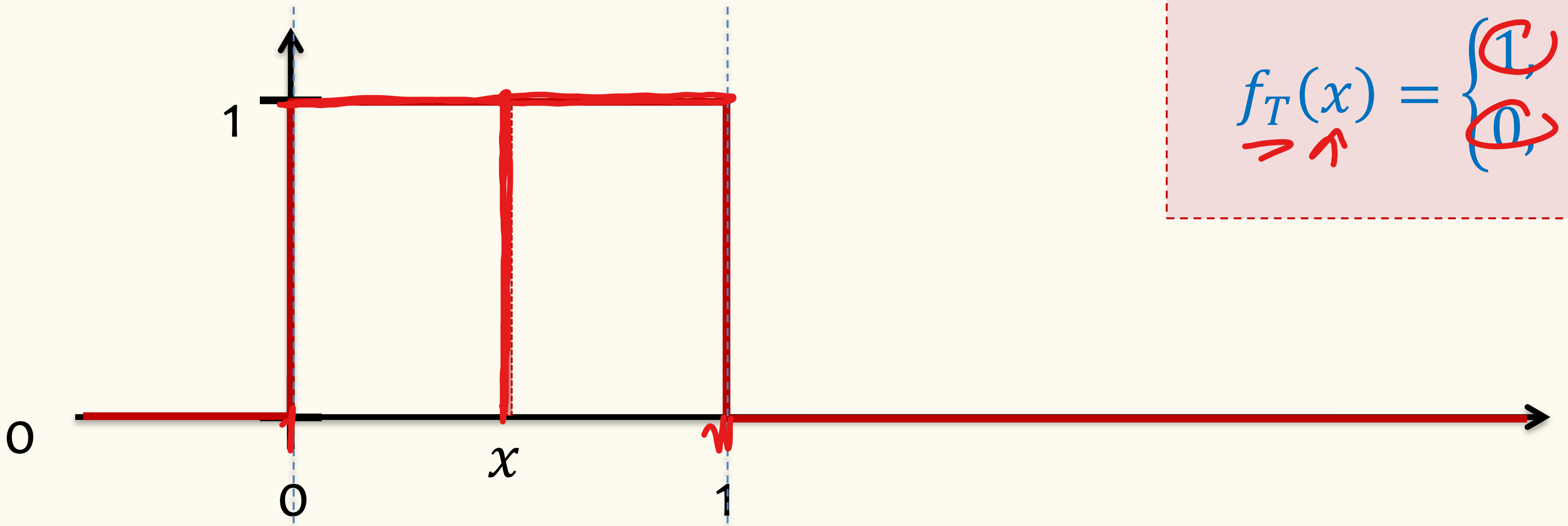
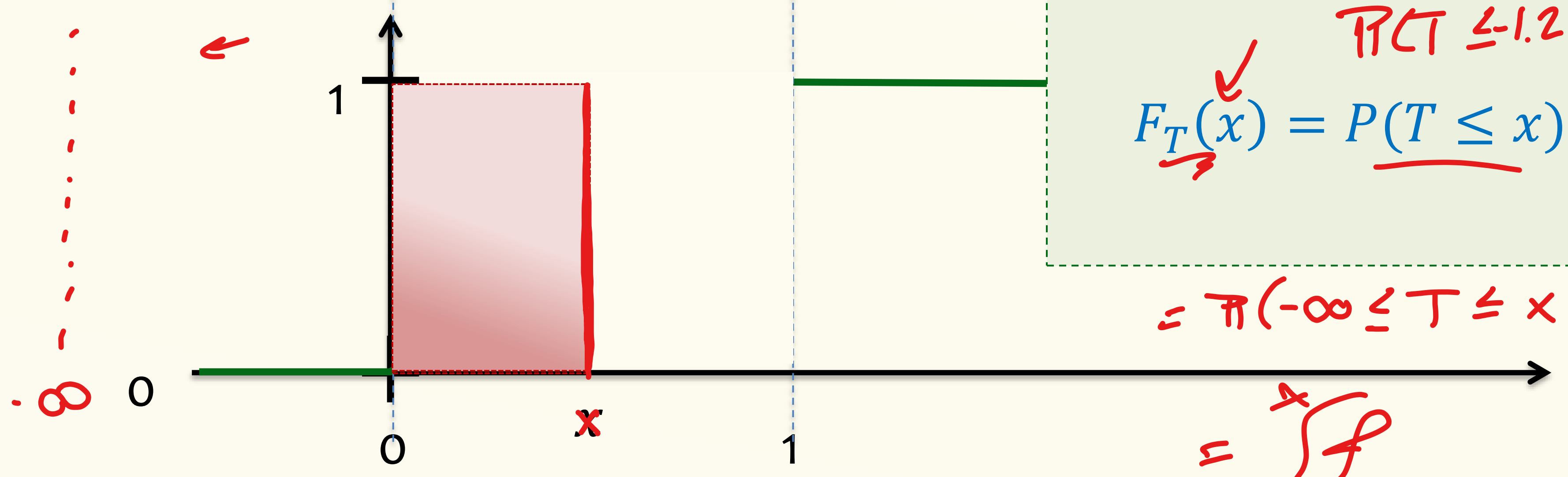
Example. $T \sim \text{Unif}(0,1)$



Probability Density Function

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



Cumulative Distribution Function

$$F_T(x) = P(T \le x) = \begin{cases} 0 & x \le 0 \\ ? & 0 \le x \le 1 \\ 1 & 1 < x \end{cases}$$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = \mathbb{P}(X \leq a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx}F(x)$

Cumulative Distribution Function

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By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F(x)$

Therefore: $\mathbb{P}(X \in [a, b]) = F(b) - F(a)$

F_X is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

$\mathrm{Lim}_{a \to -\infty} F_X(a) = P(X \leq -\infty) = 0$ $\mathrm{Lim}_{a \to +\infty} F_X(a) = P(X \leq +\infty) = 1$

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

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Expectation of a Continuous RV

Definition. The **expected value** of a continuous RV X is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Fact. $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

Definition. The **variance** of a continuous RV X is defined as

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot \big(x - \mathbb{E}(X)\big)^2 \, \mathrm{d}x = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

Uniform Distribution

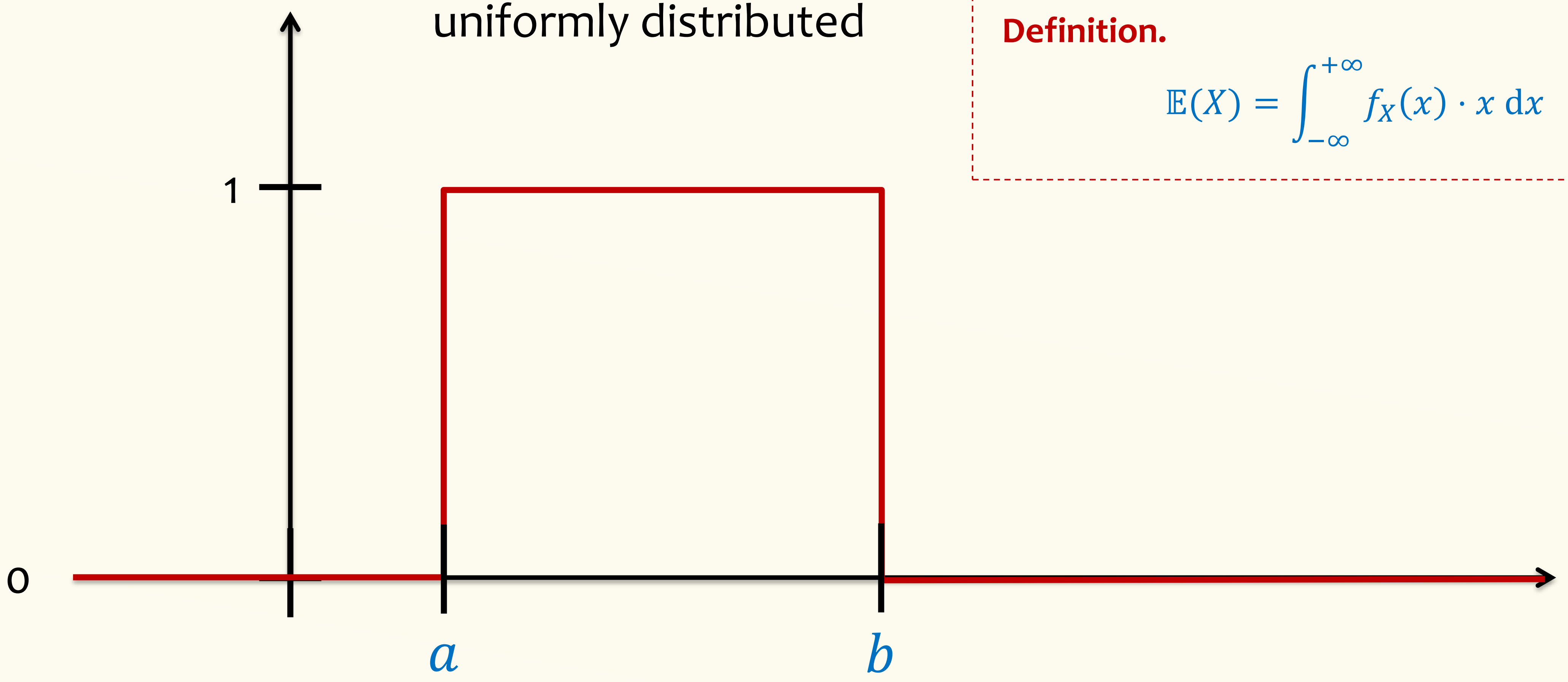
$X \sim \text{Unif}(a, b)$

We also say that X follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

Definition.

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$



Uniform Density – Expectation

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$X \sim \text{Unif}(a,b)$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

$$= \frac{1}{b-a}\int_a^b x \, \mathrm{d}x \quad = \frac{1}{b-a}\left(\frac{x^2}{2}\right)\Bigg|_a^b = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right)$$

$$= \frac{(b-a)(a+b)}{2(b-a)} \quad = \frac{a+b}{2}$$

Uniform Density – Variance

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

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$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, \mathrm{d}x$$

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$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, \mathrm{d}x$$

$$= \frac{1}{b-a}\int_a^b x^2 \, \mathrm{d}x = \frac{1}{b-a}\left(\frac{x^3}{3}\right)\Bigg|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2+ab+a^2)}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

Uniform Density – Variance

$$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}(X) = \frac{a+b}{2}$$

$X \sim \text{Unif}(a, b)$

$$\begin{aligned}
\text{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}
\end{aligned}$$

Uniform Distribution

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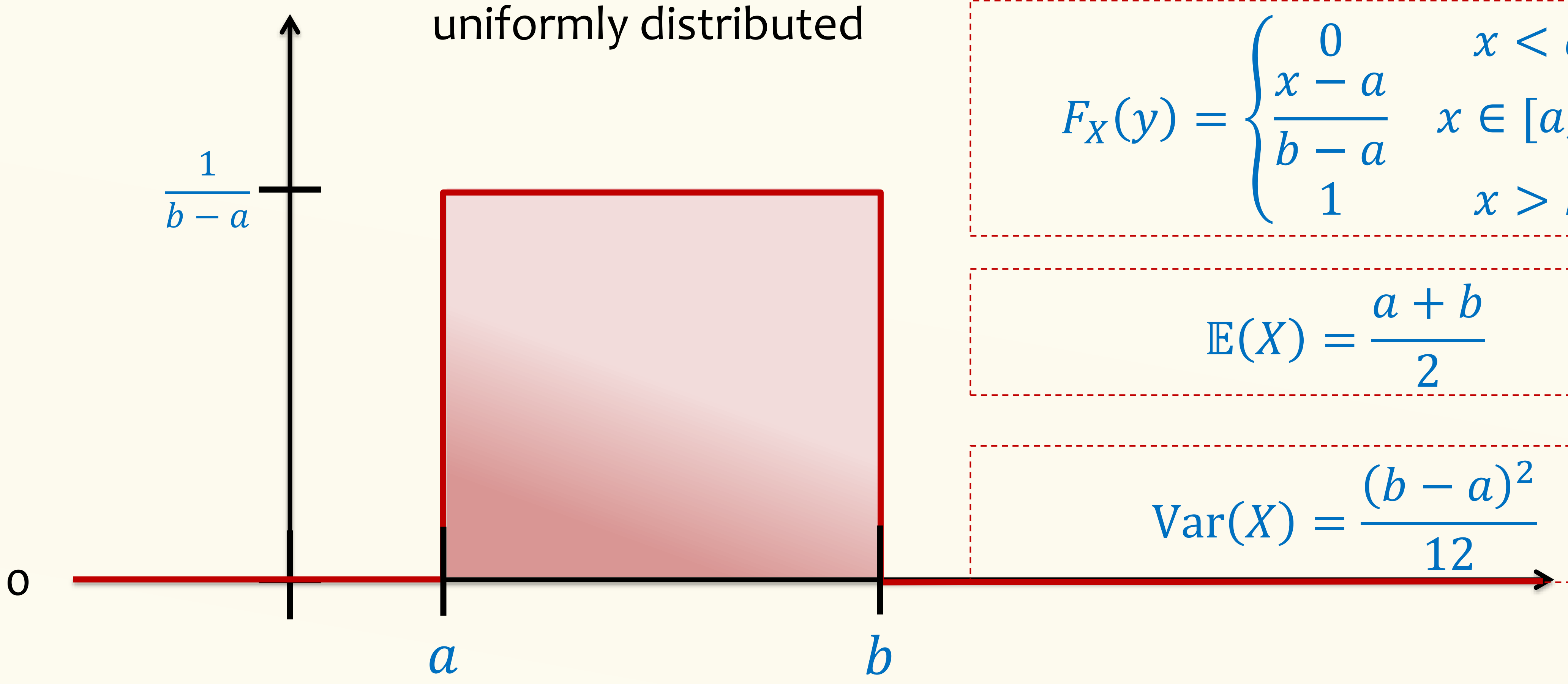


$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}(X) = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

Review – Continuous RVs

Probability Density Function (PDF).

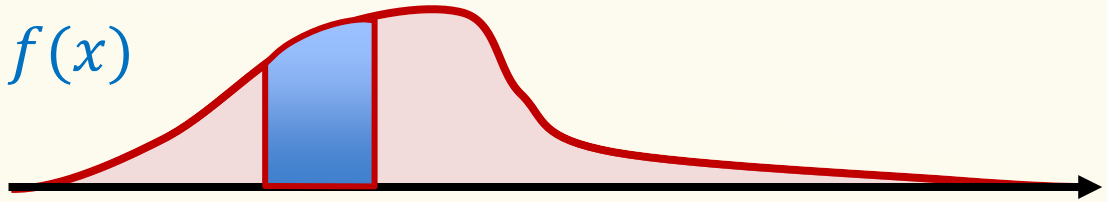
$f: \mathbb{R} \to \mathbb{R}$ s.t.

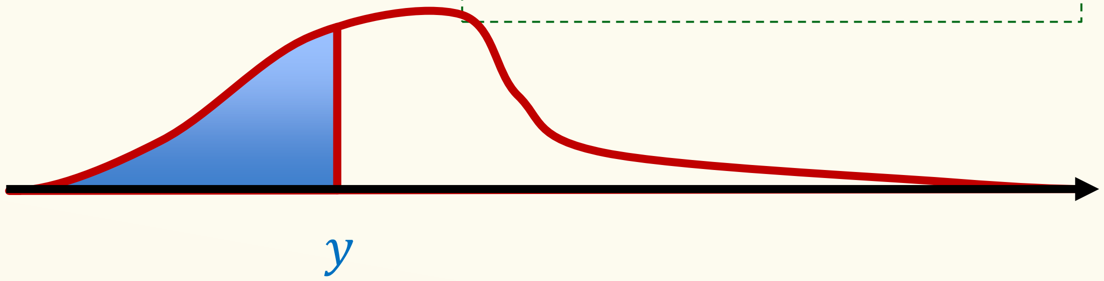
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$

Cumulative Density Function (CDF).

$$F(y) = \int_{-\infty}^{y} f(x)\,\mathrm{d}x$$

Theorem. $f(x) = \frac{dF(x)}{dx}$





Density ≠ Probability !

$$\mathbb{P}(X \in [a,b]) = \int_a^b f_X(x)\mathrm{d}x$$
$$= F_X(b) - F_X(a)$$

$$F(y) = \mathbb{P}(X \leq y)$$

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Exponential Density

Assume expected # of occurrences of an event per unit of time is λ

- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER

Numbers of occurrences of event: Poisson distribution

$$\mathbb{P}(X = i) = e^{-\lambda}\frac{\lambda^i}{i!}$$ (Discrete)

How long to wait until next event? Exponential density!

Let's define it and then derive it!

The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is λ

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0,1,2,\dots\}$
- Let $Y \sim Exp(\lambda)$ be the time till the first event. We will compute $F_Y(t)$ and $f_Y(t)$

The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is λ

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- The exponential RV has range [0, ∞], unlike Poisson with range {0,1,2,…}
- Let $Y \sim Exp(\lambda)$ be the time till the first event. We will compute $F_Y(t)$ and $f_Y(t)$
- Let $\mathrm{X} \sim Poi(t\lambda)$ be the # of events in the first t units of time, for $t \geq 0$.
- $\mathrm{P(Y > t)} = P(no\ event\ in\ the\ first\ t\ units) = P(X = 0) = e^{-t\lambda}\frac{t\lambda^0}{0!} = e^{-t\lambda}$
- $\mathrm{F_Y(t)} = 1 - P(Y > t) = 1 - e^{-t\lambda}$
- $\mathrm{f_Y(t)} = \frac{d}{dt}F_Y(t) = \lambda e^{-t\lambda}$

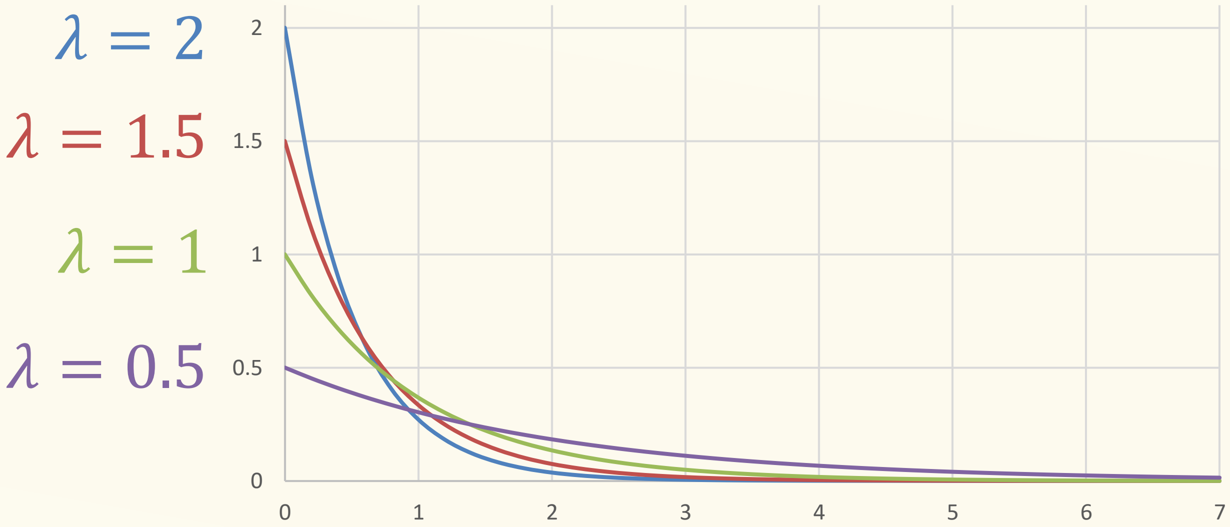
Exponential Distribution

Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$



Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \,\mathrm{d}x$$

$$= \int_{0}^{+\infty} \lambda e^{-\lambda x} \cdot x \,\mathrm{d}x$$

$$= \left(-(x + \frac{1}{\lambda})e^{-\lambda x}\right)\Bigg|_0^{\infty} = \frac{1}{\lambda}$$

Somewhat complex calculation
use integral by parts

$$\mathbb{E}(X) = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins.

Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?

$$T \sim Exp(\frac{1}{10})$$

$$P(10 \leq T \leq 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} dx$$

$$y = \frac{x}{10}, dy = \frac{dx}{10}$$

$$P(10 \leq T \leq 20) = \int_{1}^{2} e^{-y} dy = -e^{-y} \Big|_1^2 = e^{-1} - e^{-2}$$

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Memorylessness

Definition. A random variable is **memoryless** if for all $s, t > 0$,

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Assuming exp distr, if you've waited s minutes,
prob of waiting t more is exactly same as $s = 0$

Memorylessness of Exponential

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Proof.

$$\mathbb{P}(X > s + t \mid X > s)$$

Memorylessness of Exponential

Assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as $s = 0$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$$\mathbb{P}(X > s + t \mid X > s) = \frac{\mathbb{P}(\{X > s + t\} \cap \{X > s\})}{\mathbb{P}(X > s)}$$

$$= \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t)$$

The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous)