CSE 312

# Foundations of Computing II

## Lecture 12: Continuous Random Variables

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

# Agenda

$X \sim Bin(10, 0.7)$

- Poisson RV
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Expectation and Variance

# Poisson Distribution

Siméon Denis Poisson
1781-1840

- Suppose "events" happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. *with parameter* λ (denoted X ~ Poi(λ)) and has distribution (PMF):

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Several examples of "Poisson processes":

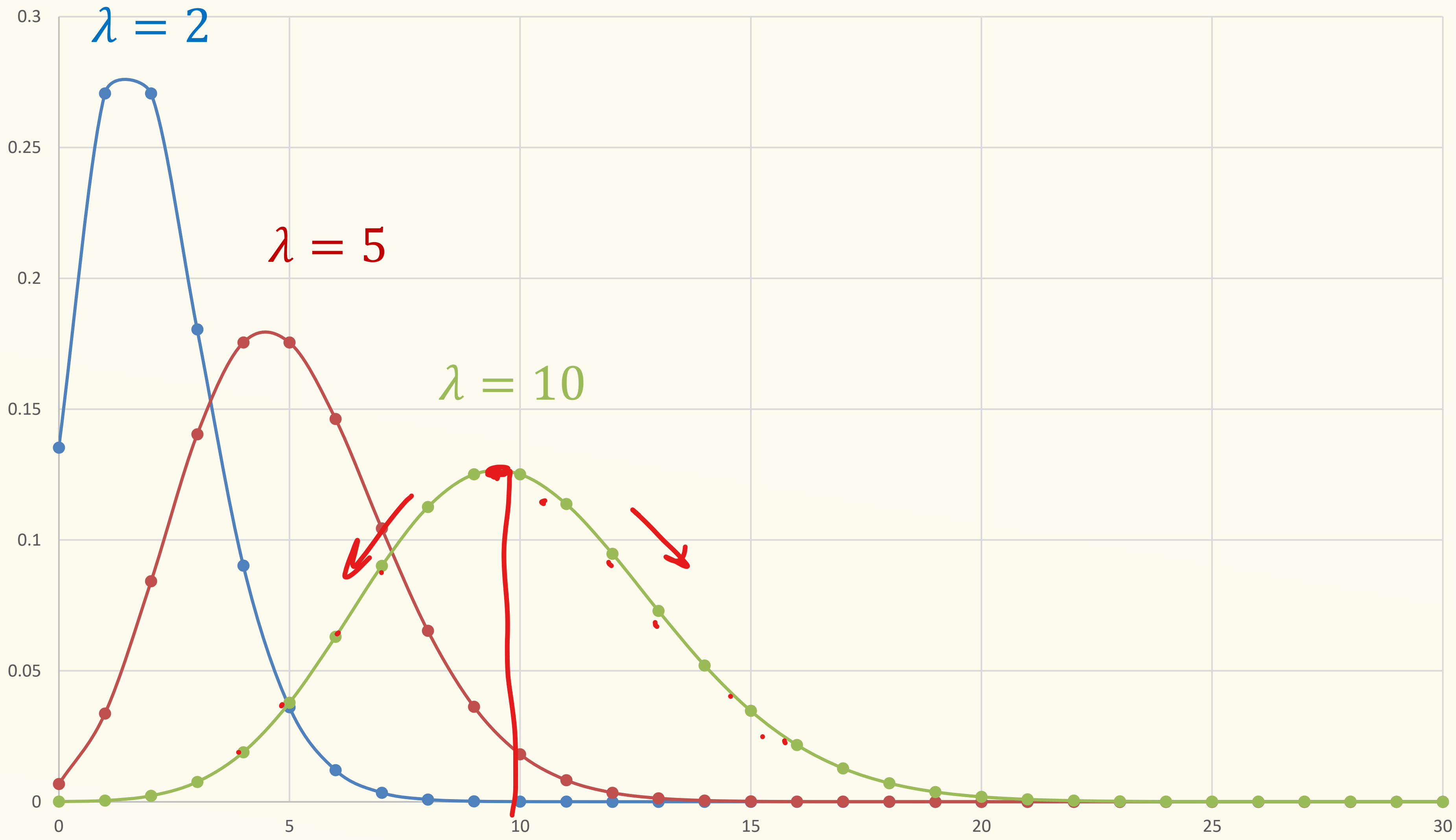
- # of cars passing through a certain town in 1 hour
- # of requests to web servers in a minute
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume fixed average rate

# Probability Mass Function

$[0, \infty)$

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



# Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) =$$

**Fact.** $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$

# Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$$

**Fact.** $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$

# Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If $X$ is a Poisson RV with parameter $\lambda$, then

$$\mathbb{E}(X) = \lambda$$

**Proof.** $\mathbb{E}(X) = \sum_{i=0}^{\infty} i \cdot \mathbb{P}(X = i)$

# Expectation

Poi($\lambda$)

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If $X$ is a Poisson RV with parameter $\lambda$, then

$$\mathbb{E}(X) = \lambda$$

**Proof.**

$$\mathbb{E}(X) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

= 1 (see prior slides!)

$$= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \boxed{\lambda}$$

# Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If $X$ is a Poisson RV with parameter $\lambda$, then $\mathrm{Var}(X) = \lambda$

**Proof.** $\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 \quad = \lambda^2 + \lambda$

$$\Rightarrow \quad \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

# Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If $X$ is a Poisson RV with parameter $\lambda$, then $\mathrm{Var}(X) = \lambda$

**Proof.** $$\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1)$$

$$= \lambda \left[ \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}(X) = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda$$

Similar to the previous proof
Verify offline.

$$\Rightarrow \quad \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

# Poisson Random Variables

**Definition.** A **Poisson random variable** $X$ with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Notation:** $X \sim \mathrm{Poi}(\lambda)$

**PMF:** $\Pr(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

**Expectation:** $\mathrm{E}[X] = \lambda$

**Variance:** $\mathrm{Var}(X) = \lambda$

# Sum of Independent Poisson RVs

**Theorem.** Let $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $\mathrm{Z} = (X + Y)$. For all $k = 0,1,2,3 \ldots,$

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

More generally, let $X_1 \sim Poi(\lambda_1), \cdots, X_n \sim Poi(\lambda_n)$ such that $\lambda = \Sigma_i \lambda_i$.
Let $\mathrm{Z} = \Sigma_i X_i$

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

# Poisson Example

There are two ERs in a small town that act independently. The first has an average of 4 patients admitted per hour, and the second has an average of 3. What is the likelihood that in the next hour, 10 patients are admitted across both ERs?

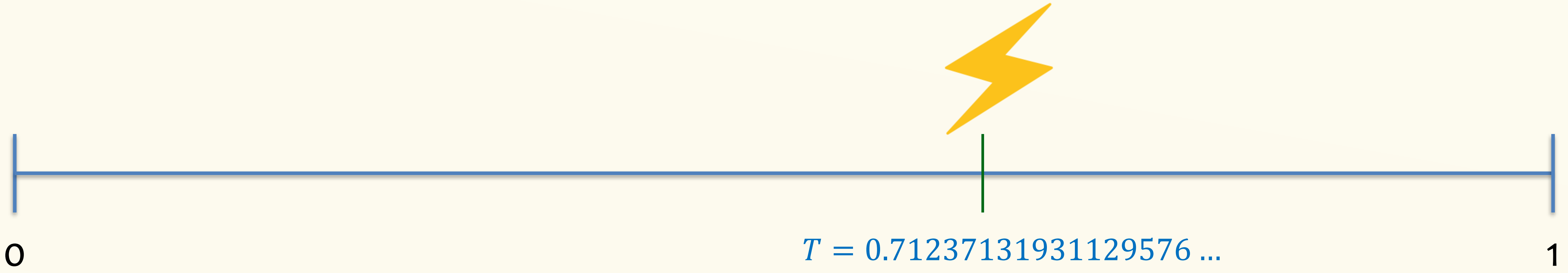
# Agenda

- Poisson RV
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Expectation and Variance

# Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

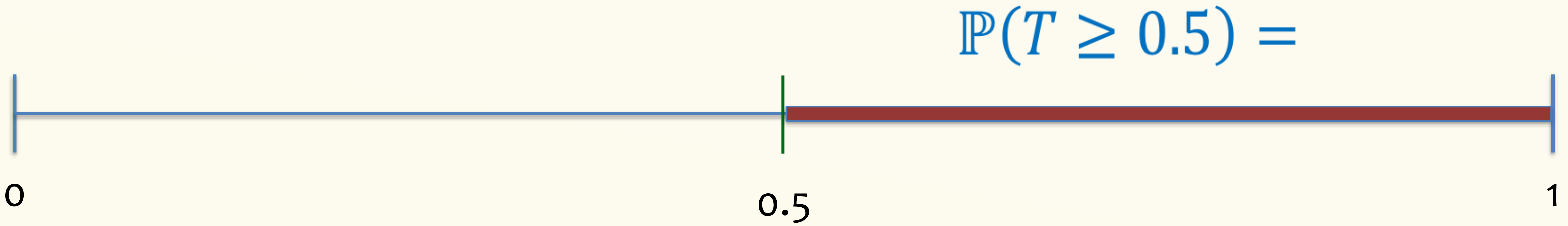
- $T$ = time of lightning strike
- Every time within $[0,1]$ is equally likely
  - – Time measured with infinitesimal precision.




The outcome space is not discrete

# Lightning strikes a pole within a one-minute time frame

- $T$ = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



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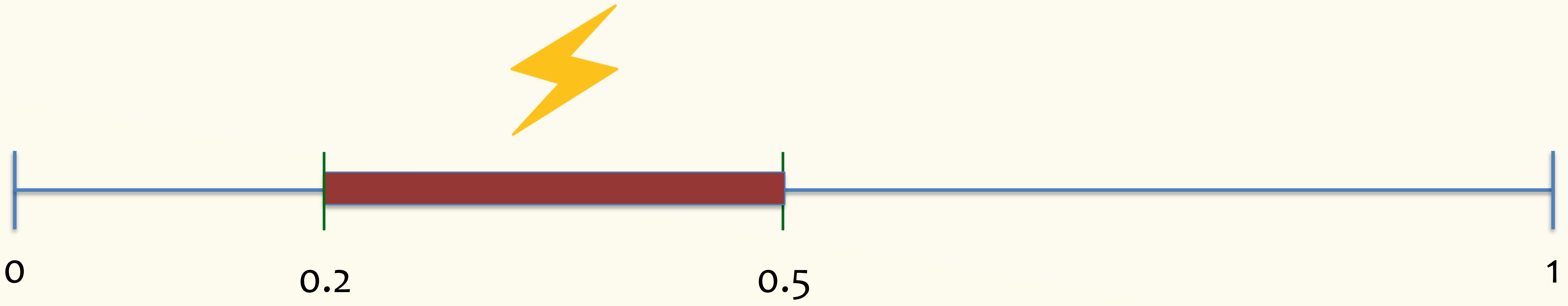


$$\mathbb{P}(0.2 \leq T \leq 0.5) =$$

Lightning strikes a pole within a one-minute time frame

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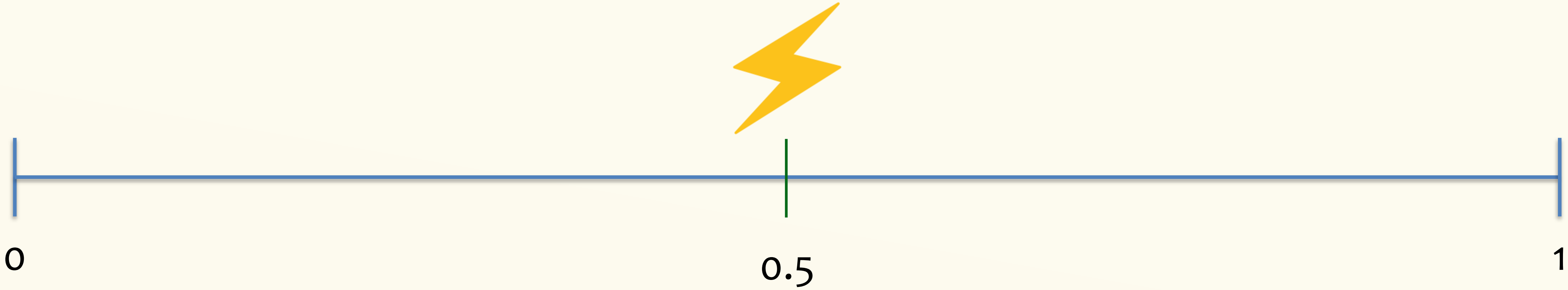


$\mathbb{P}(T = 0.5) =$

# Bottom line

- This gives rise to a different type of random variable
- $\mathbb{P}(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
  - $\mathbb{P}(T \in [0,1]) = 1$
  - $\mathbb{P}(T \in [a, b]) = b - a$
  - …
- How do we model the behavior of $T$?

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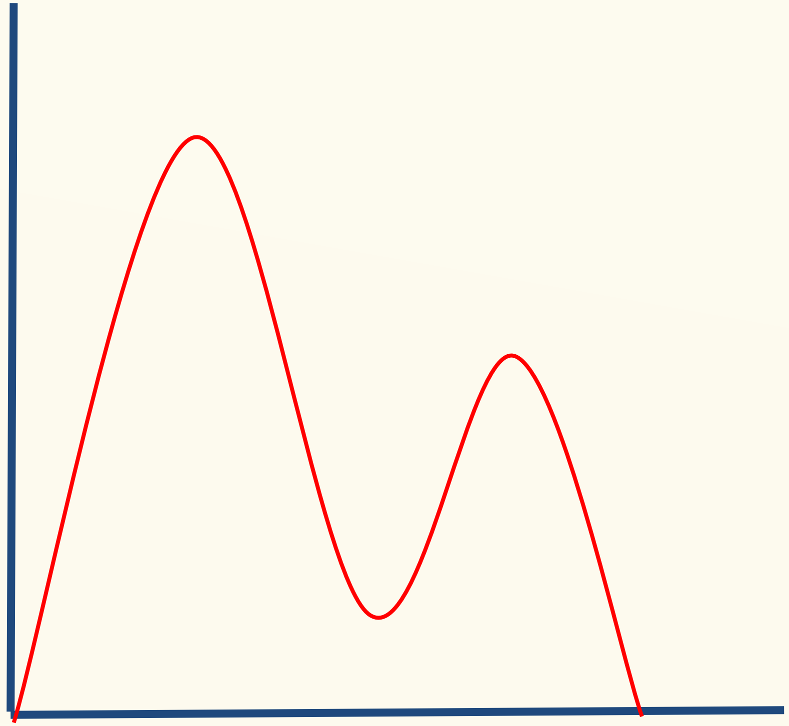
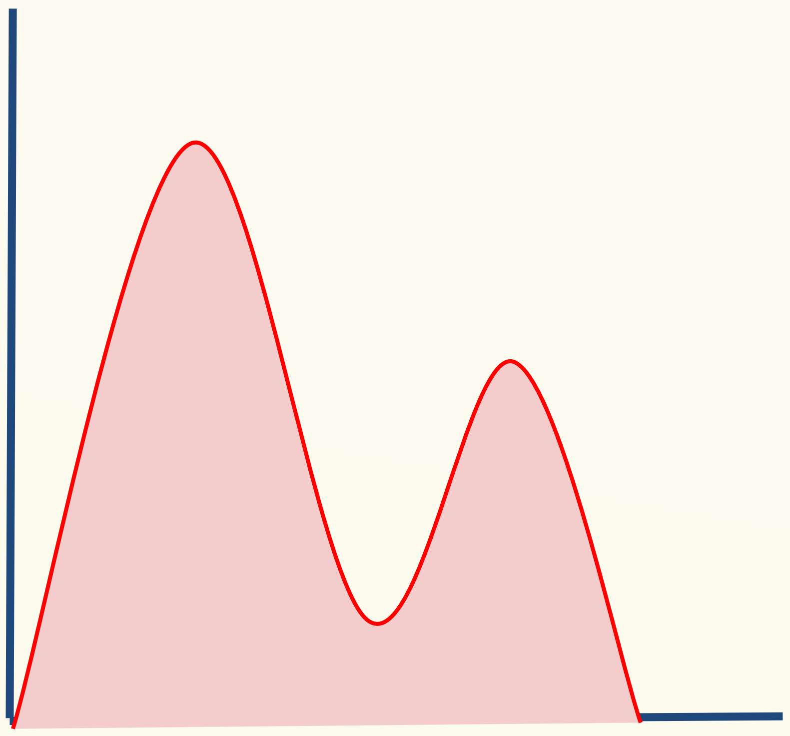
**Definition.** A **continuous random variable** $X$ is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \to \mathbb{R}$, such that



**Non-negativity:** $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

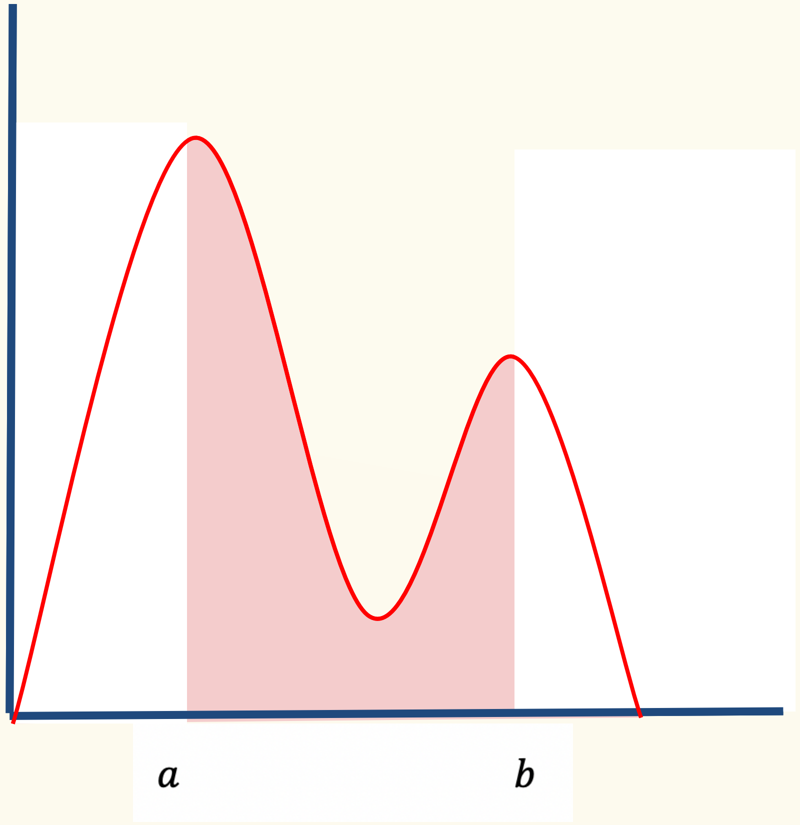
# Probability Density Function - Intuition



**Non-negativity:** $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

**Normalization:** $\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = 1$

# Probability Density Function - Intuition

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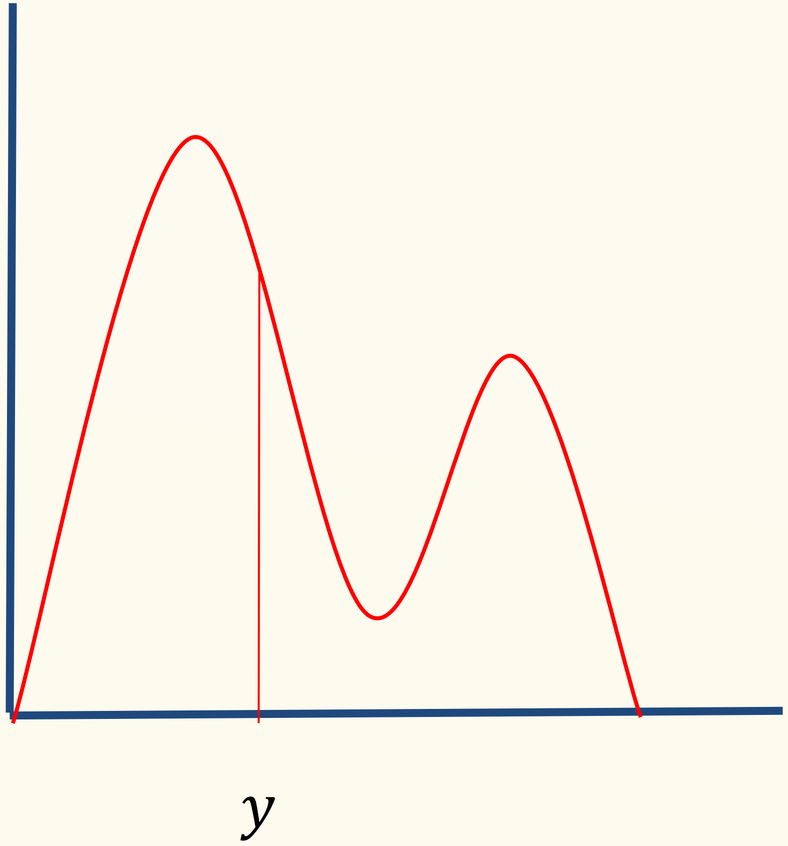
$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

# Probability Density Function - Intuition




**Non-negativity:** $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

**Normalization:** $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

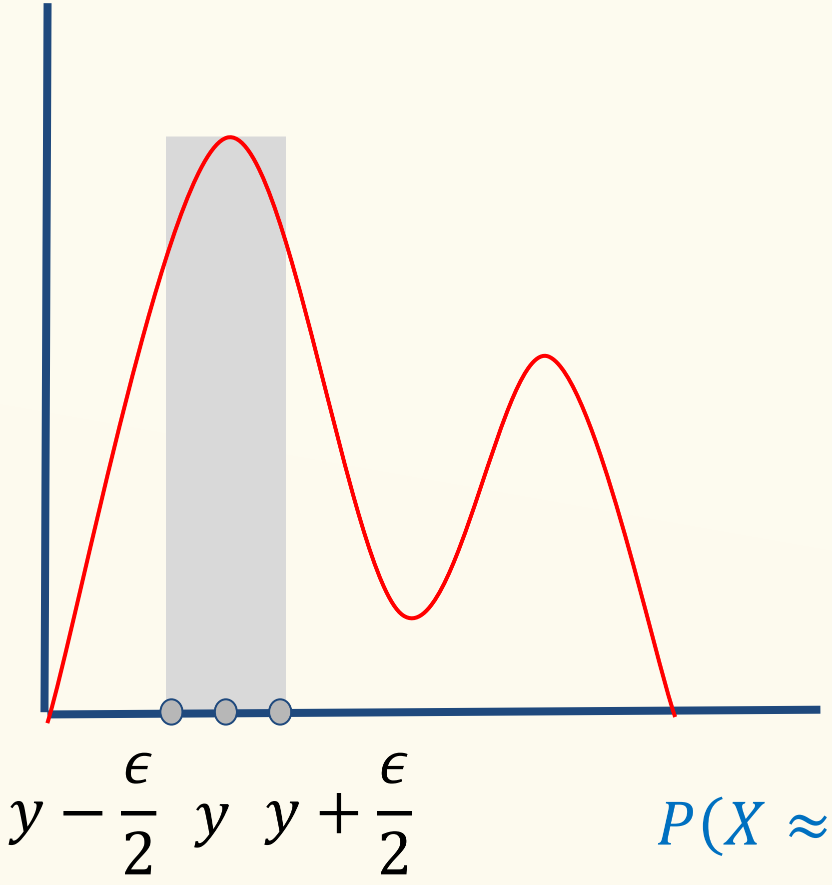
$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

**Density ≠ Probability**

$f_X(y) \neq 0 \qquad \mathbb{P}(X = y) = 0$

# Probability Density Function - Intuition

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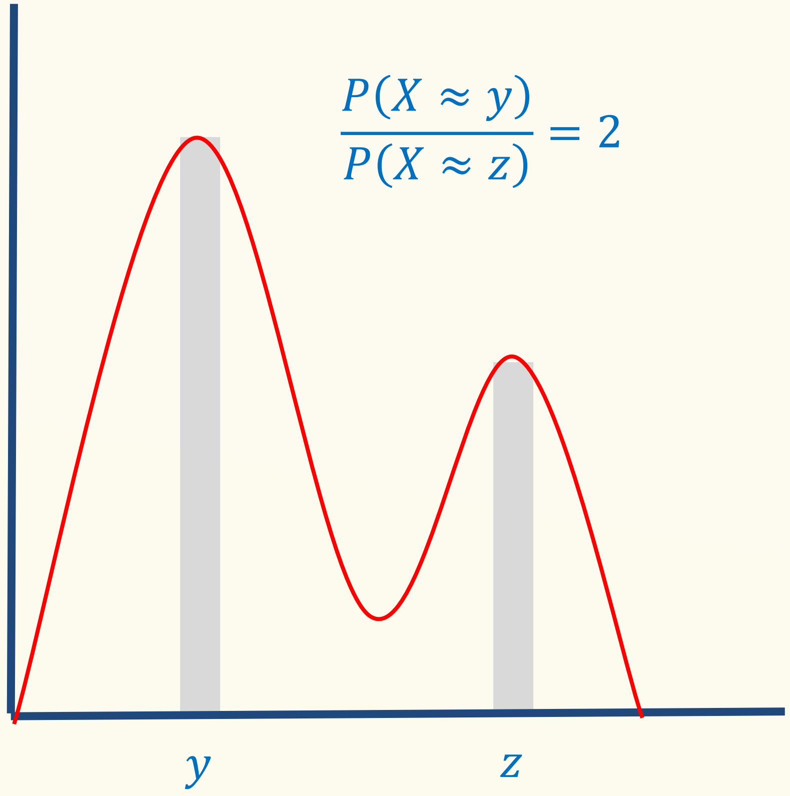
$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

# Probability Density Function - Intuition

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$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

**Definition.** A **continuous random variable** $X$ is defined by a **probability density function** (PDF) $f_X \colon \mathbb{R} \to \mathbb{R}$, such that

**Non-negativity:** $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

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$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

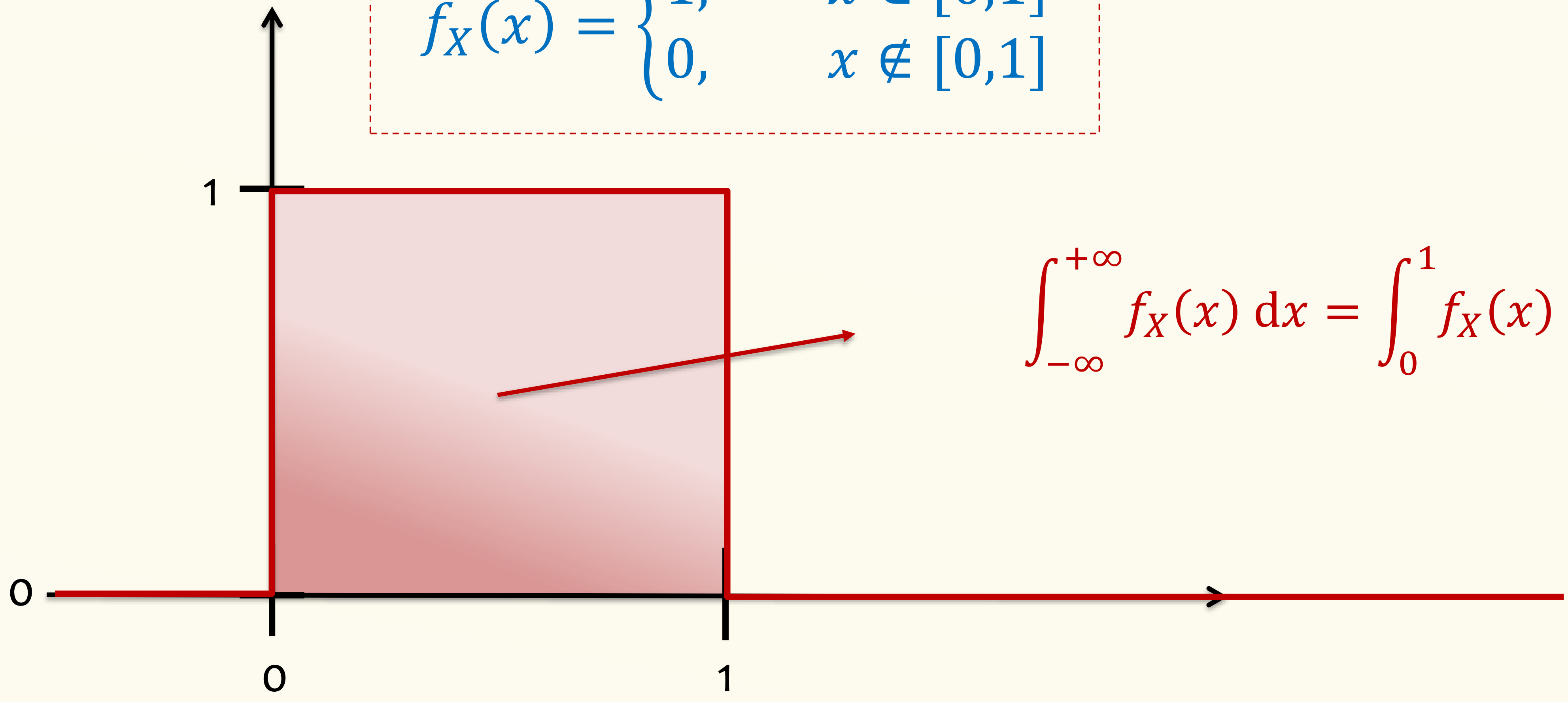
# PDF of Uniform RV

$X \sim \text{Unif}(0,1)$

**Non-negativity:** $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

**Normalization:** $\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = 1$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$




$$\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = \int_0^1 f_X(x)\, \mathrm{d}x = 1 \cdot 1 = 1$$

# Probability of Event

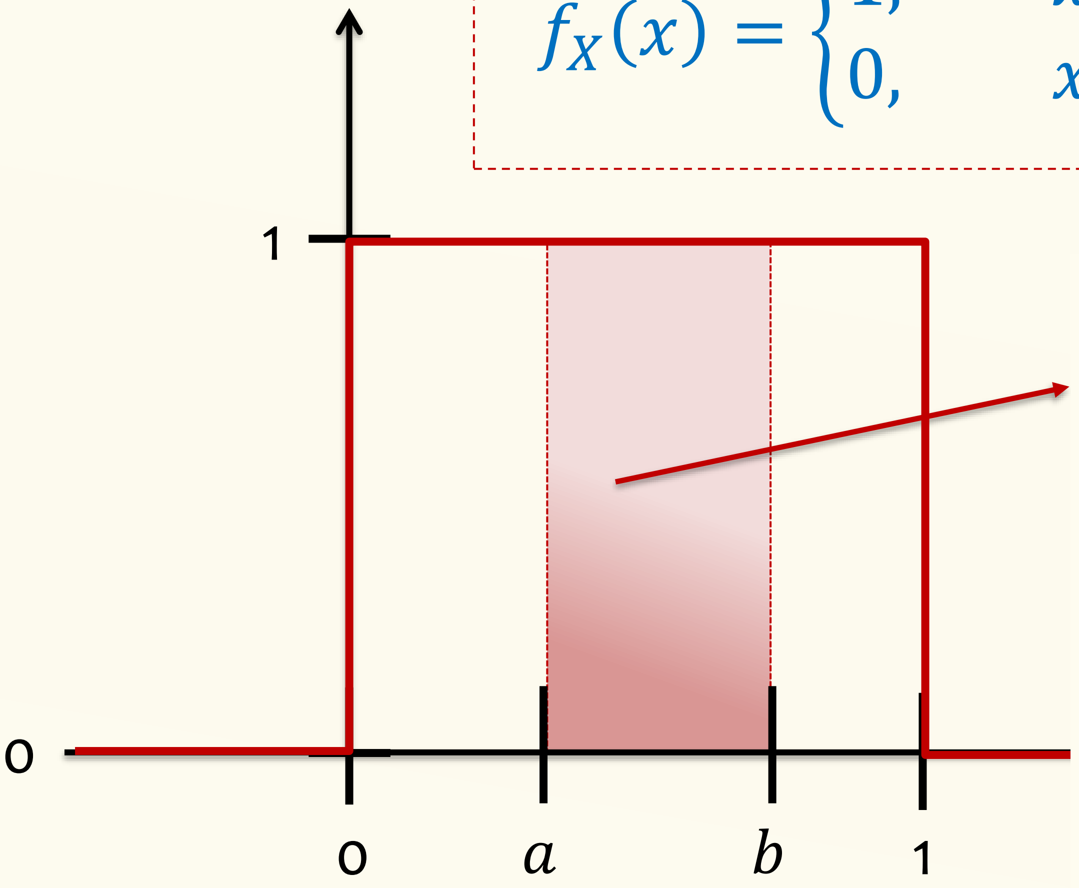
$X \sim \text{Unif}(0,1)$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

**Non-negativity:** $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

**Normalization:** $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$




1. If $0 \leq a \; and \; b \leq 1$
   $\mathbb{P}(a \leq X \leq b) = b - a$
2. If $a < 0 \; and \; 0 \leq b \leq 1$
   $\mathbb{P}(a \leq X \leq b) = b$
3. If $a \geq 0 \; and \; b > 1$
   $\mathbb{P}(a \leq X \leq b) = b - a$
4. If $a < 0 \; and \; b > 1$
   $\mathbb{P}(a \leq X \leq b) = 1$

A. All of them are correct
B. Only 1, 2, 4 are right
C. Only 1 is right
D. Only 1 and 2 are right

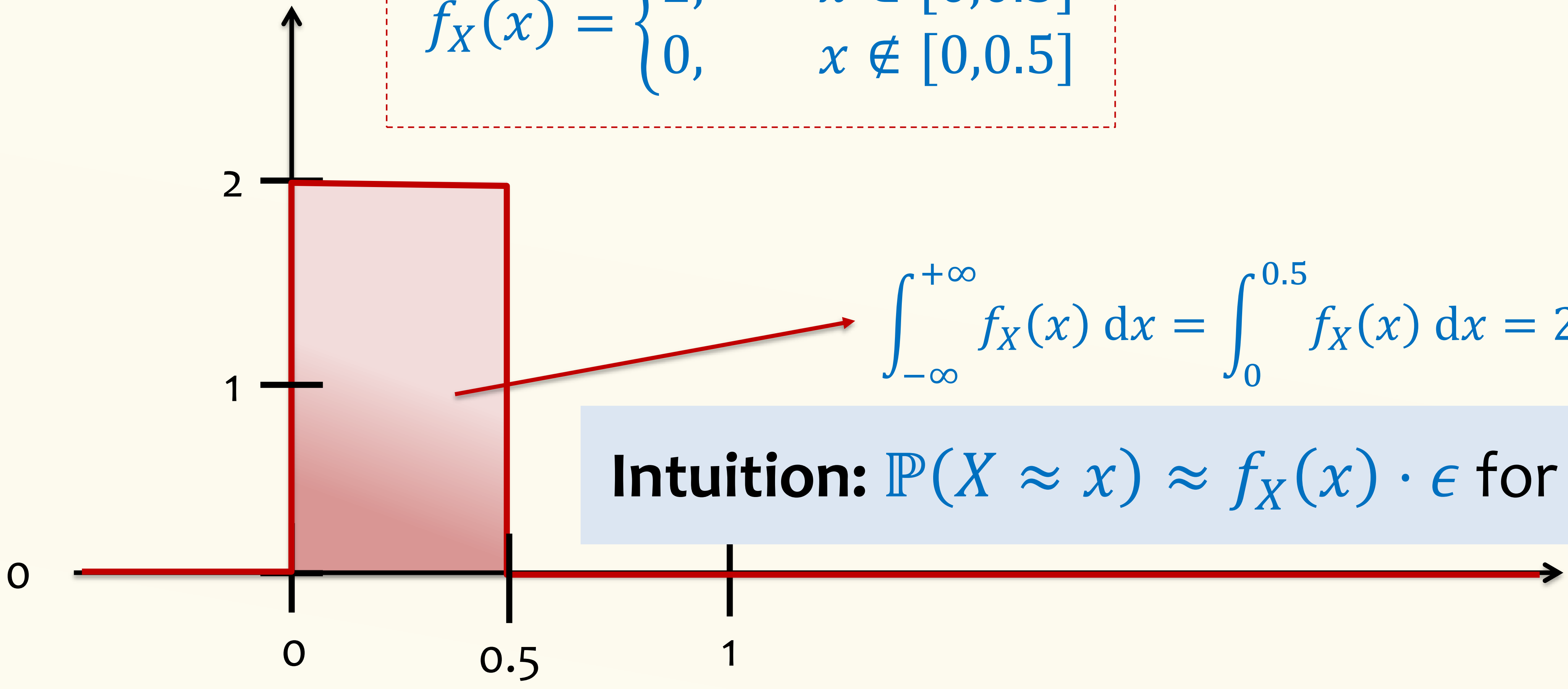
# PDF of Uniform RV

**Density $\neq$ Probability**

$f_X(x) \gg 1$ is possible!

$X \sim \text{Unif}(0,0.5)$

$$f_X(x) = \begin{cases} 2, & x \in [0,0.5] \\ 0, & x \notin [0,0.5] \end{cases}$$



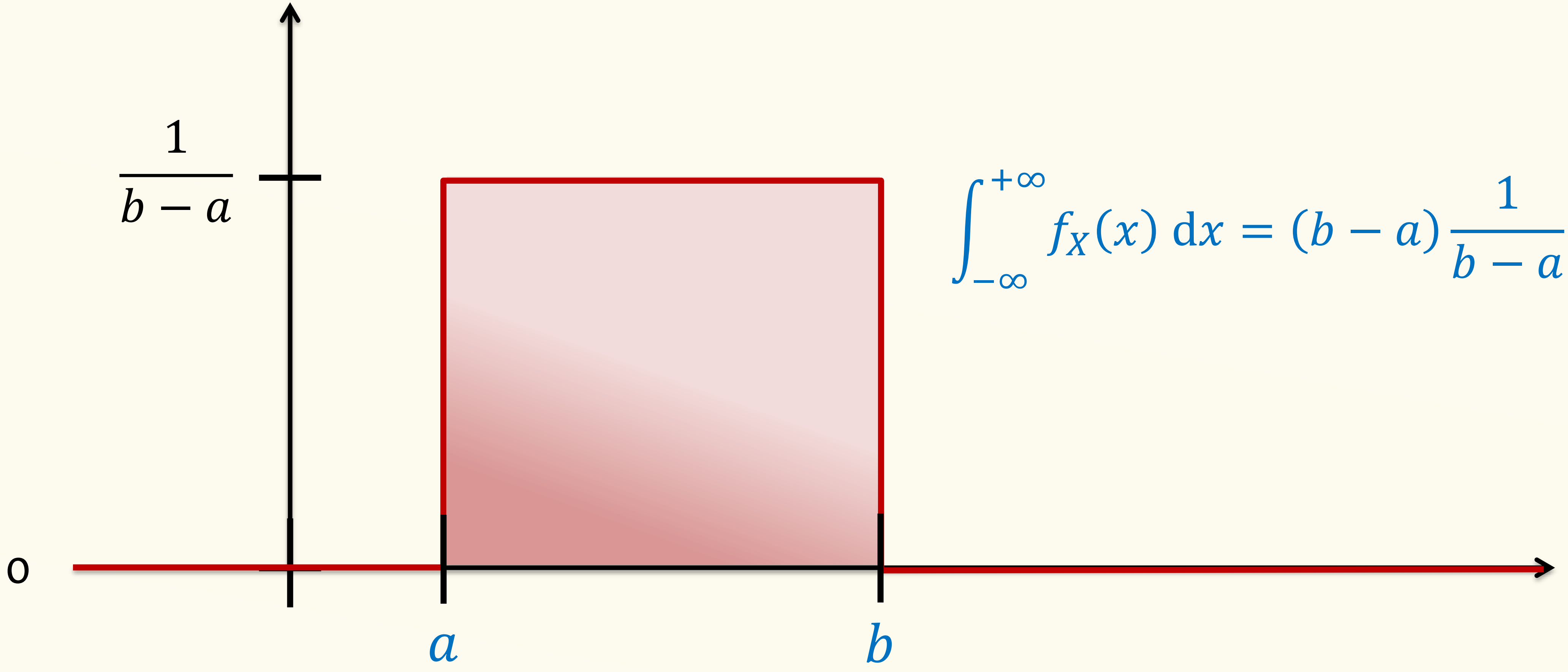

$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^{0.5} f_X(x)\,\mathrm{d}x = 2 \cdot 0.5 = 1$$

**Intuition:** $\mathbb{P}(X \approx x) \approx f_X(x) \cdot \epsilon$ for small $\epsilon$

# Uniform Distribution

$$X \sim \text{Unif}(a, b)$$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$




$$\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = (b-a)\frac{1}{b-a} = 1$$

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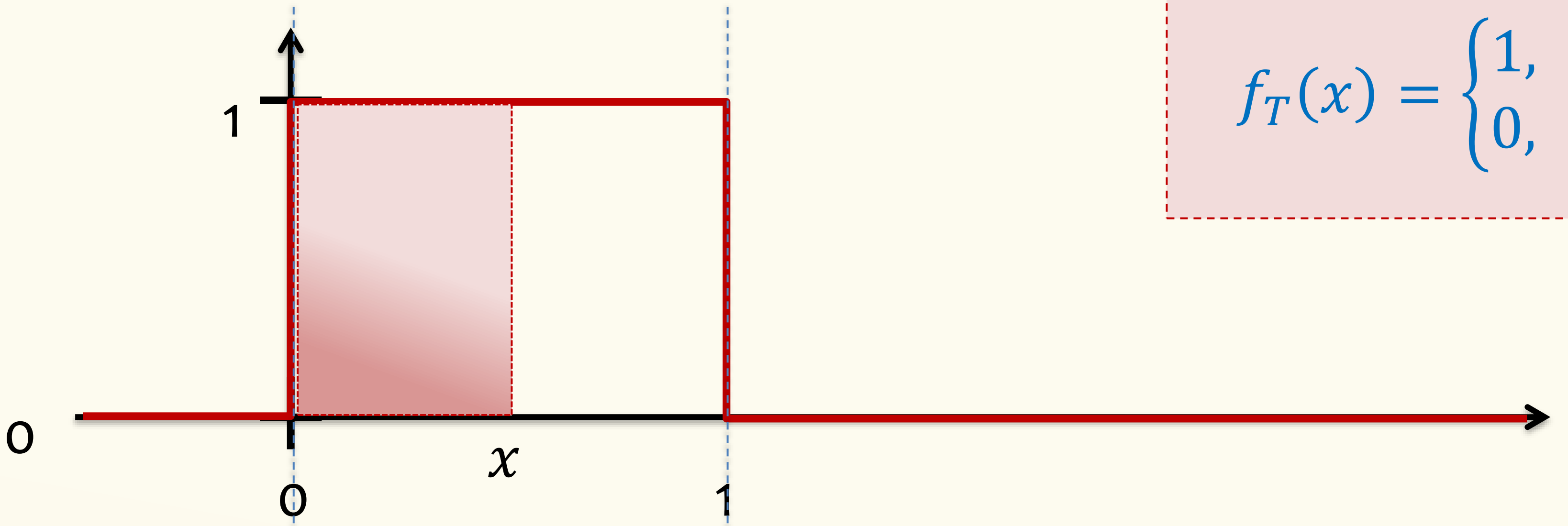
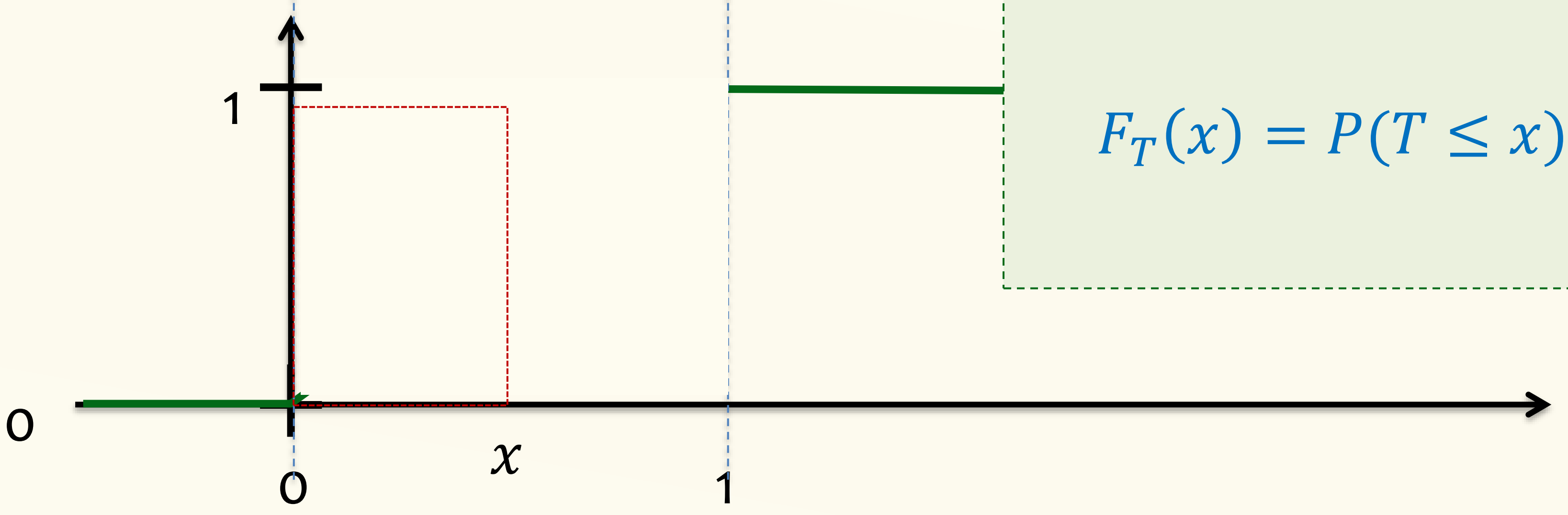
- Expectation and Variance

**Example.** $T \sim \text{Unif}(0,1)$




**Probability Density Function**

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$




**Cumulative Distribution Function**

$$F_T(x) = P(T \le x) = \begin{cases} 0 & x \le 0 \\ ? & 0 \le x \le 1 \\ 1 & 1 \le x \end{cases}$$

# Cumulative Distribution Function

**Definition.** The **cumulative distribution function (cdf)** of $X$ is

$$F_X(a) = \mathbb{P}(X \leq a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx}F(x)$

# Cumulative Distribution Function

**Definition.** The **cumulative distribution function (cdf)** of $X$ is

$$F_X(a) = \mathbb{P}(X \le a) = \int_{-\infty}^{a} f_X(x)\,\mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx}F(x)$

Therefore: $\mathbb{P}(X \in [a, b]) = F(b) - F(a)$

$F_X$ is monotone increasing, since $f_X(x) \ge 0$. That is $F_X(c) \le F_X(d)$ for $c \le d$

$\mathrm{Lim}_{a \to -\infty} F_X(a) = P(X \le -\infty) = 0$ $\mathrm{Lim}_{a \to +\infty} F_X(a) = P(X \le +\infty) = 1$

# From Discrete to Continuous

| | Discrete | Continuous |
|---|---|---|
| **PMF/PDF** | $p_X(x) = P(X = x)$ | $f_X(x) \neq P(X = x) = 0$ |
| **CDF** | $F_X(x) = \sum_{t \leq x} p_X(t)$ | $F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$ |
| **Normalization** | $\sum_x p_X(x) = 1$ | $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$ |
| **Expectation** | $\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$ | $\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$ |

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# Expectation of a Continuous RV

**Definition.** The **expected value** of a continuous RV $X$ is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

**Fact.** $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

**Definition.** The **variance** of a continuous RV $X$ is defined as

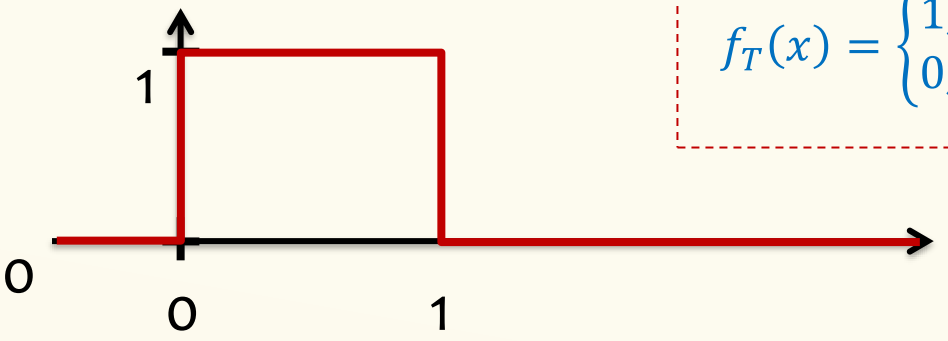
$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot \left(x - \mathbb{E}(X)\right)^2 \mathrm{d}x = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

# Expectation of a Continuous RV

**Definition.**

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

**Example.** $T \sim \text{Unif}(0,1)$



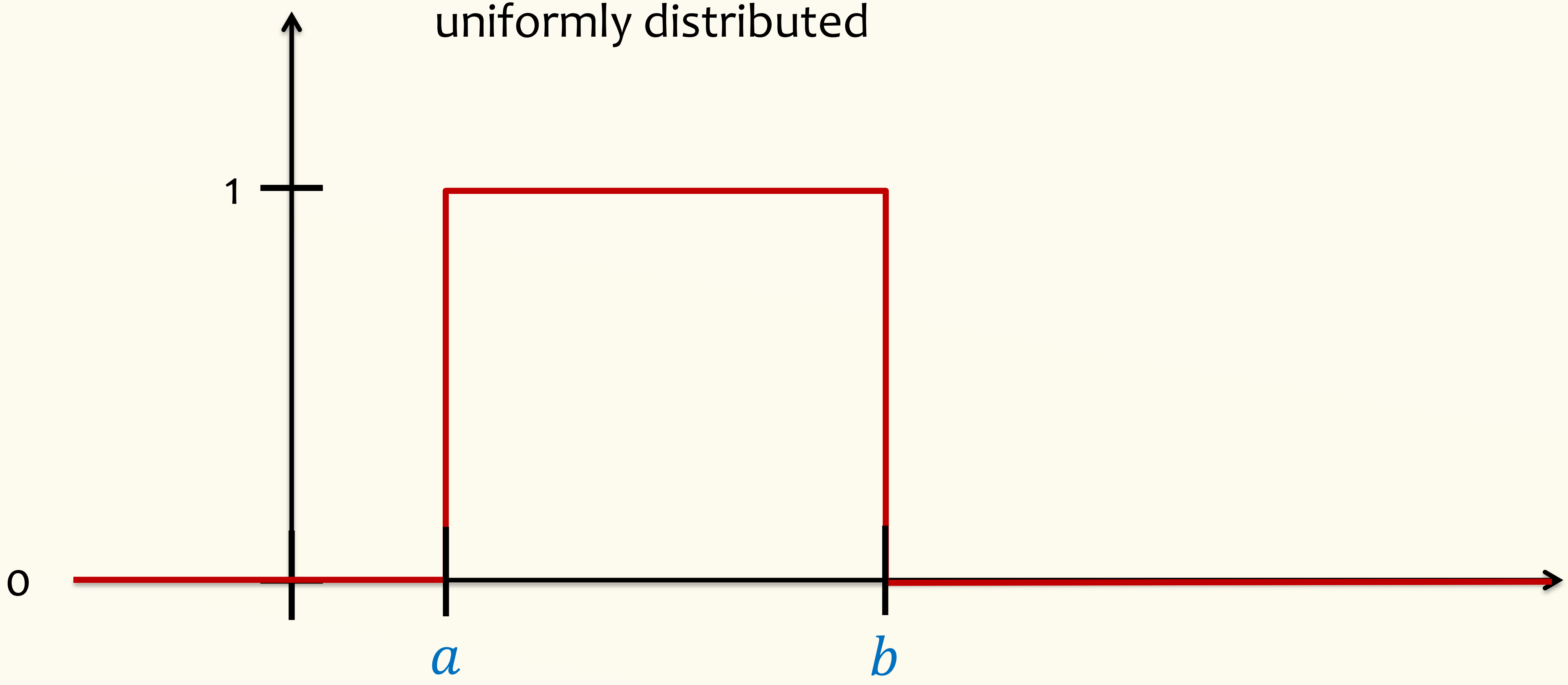

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

# Uniform Distribution

$X \sim \text{Unif}(a, b)$

We also say that $X$ follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



# Uniform Density – Expectation

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a, b)$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

# Uniform Density – Expectation

$$X \sim \mathrm{Unif}(a, b)$$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

$$= \frac{1}{b-a} \int_a^b x \, \mathrm{d}x \quad = \frac{1}{b-a} \left( \frac{x^2}{2} \right) \Big|_a^b = \frac{1}{b-a} \left( \frac{b^2 - a^2}{2} \right)$$

$$= \frac{(b-a)(a+b)}{2(b-a)} \quad = \frac{a+b}{2}$$

# Uniform Density – Variance

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \text{Unif}(a,b)$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, \mathrm{d}x$$

# Uniform Density – Variance

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \text{Unif}(a,b)$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, \mathrm{d}x$$

$$= \frac{1}{b-a}\int_a^b x^2 \, \mathrm{d}x = \frac{1}{b-a}\left(\frac{x^3}{3}\right)\Bigg|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

# Uniform Density – Variance

$$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}(X) = \frac{a+b}{2}$$

$X \sim \text{Unif}(a, b)$

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$