

CSE 312

Foundations of Computing II


Lecture 12: Continuous Random Variables



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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊

Agenda

- Poisson RV 
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function
- Expectation and Variance

Poisson Distribution

- Suppose “events” happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. with parameter λ (denoted $X \sim \text{Poi}(\lambda)$) and has distribution (PMF):

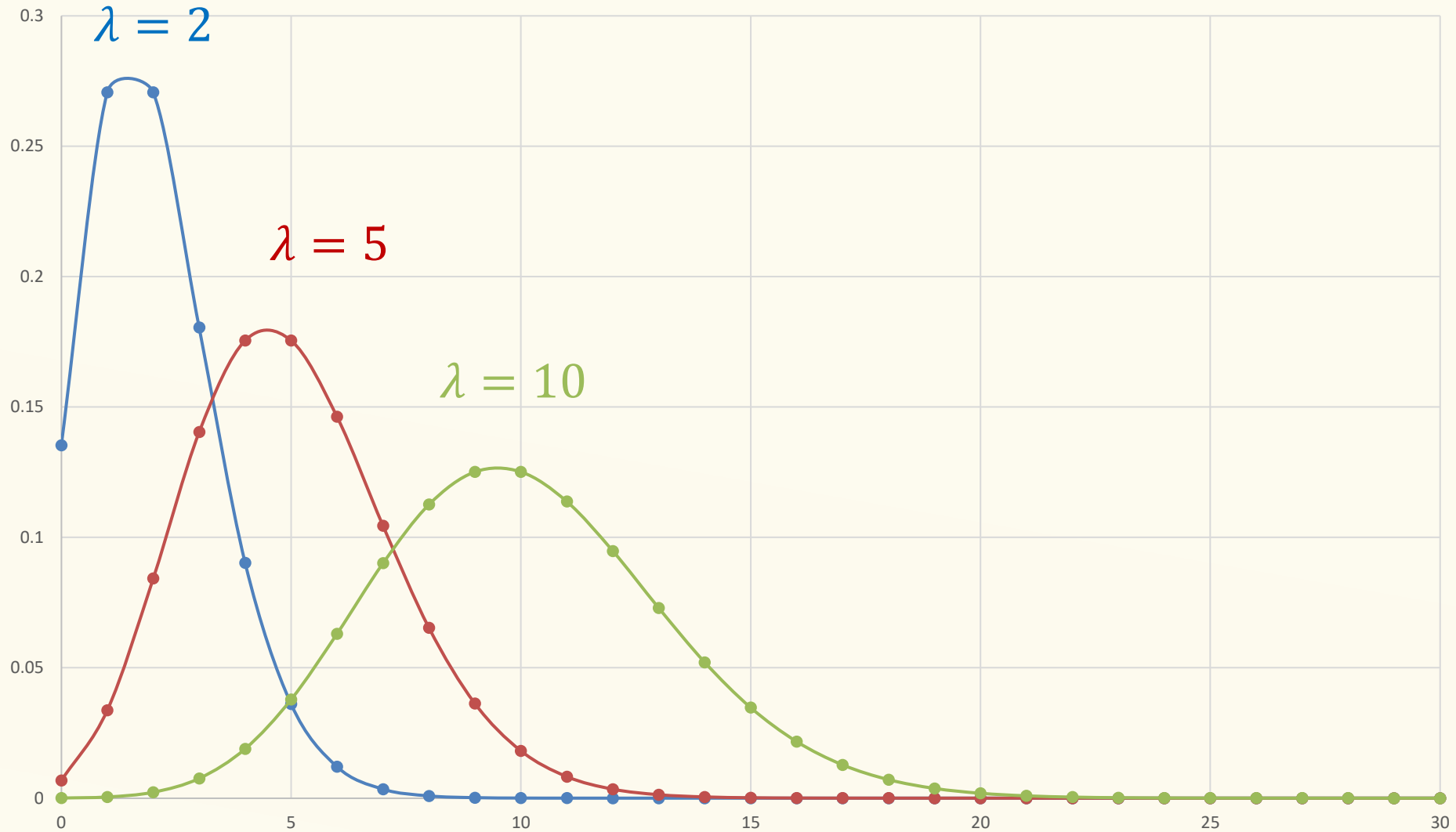
$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Several examples of “Poisson processes”:

- # of cars passing through a certain town in 1 hour
 - # of requests to web servers in a minute
 - # of photons hitting a light detector in a given interval
 - # of patients arriving to ER within an hour
- Assume fixed average rate

Probability Mass Function

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) =$$

$$\text{Fact. } \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$$

Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}_{= e^{\lambda}} = e^{-\lambda} e^{\lambda} = 1$$

$$\text{Fact. } \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$$

Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ , then

$$\mathbb{E}(X) = \lambda$$

Proof. $\mathbb{E}(X) = \sum_{i=0}^{\infty} i \cdot \mathbb{P}(X = i)$

Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ , then

$$\mathbb{E}(X) = \lambda$$

Proof.

$$\begin{aligned}\mathbb{E}(X) &= \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} \\ &= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \\ &= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda\end{aligned}$$

= 1 (see prior slides!)

Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ , then $\text{Var}(X) = \lambda$

Proof.
$$\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \lambda^2 + \lambda$$



$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ , then $\text{Var}(X) = \lambda$

Proof.

$$\begin{aligned}\mathbb{E}(X^2) &= \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i \\ &= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1) \\ &= \lambda \left[\underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}(X) = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda\end{aligned}$$

Similar to the previous proof
Verify offline.



$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0, 1, 2, 3 \dots$,

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Notation: $X \sim \text{Poi}(\lambda)$

PMF: $\Pr(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

Expectation: $E[X] = \lambda$

Variance: $\text{Var}(X) = \lambda$

Sum of Independent Poisson RVs

Theorem. Let $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = (X + Y)$. For all $k = 0, 1, 2, 3, \dots$,

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

More generally, let $X_1 \sim Poi(\lambda_1), \dots, X_n \sim Poi(\lambda_n)$ such that $\lambda = \sum_i \lambda_i$.


Let $Z = \sum_i X_i$

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Poisson Example

There are two ERs in a small town that act independently. The first has an average of 4 patients admitted per hour, and the second has an average of 3. What is the likelihood that in the next hour, 10 patients are admitted across both ERs?

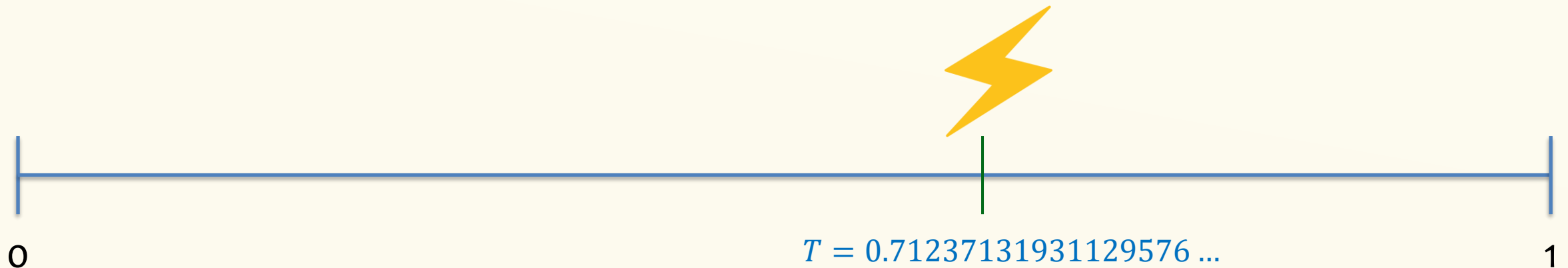
Agenda

- Poisson RV
- Continuous Random Variables 
- Probability Density Function
- Cumulative Distribution Function
- Expectation and Variance

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

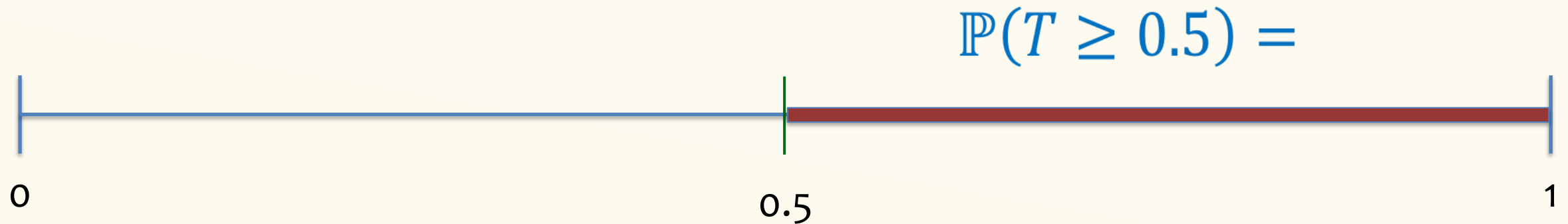
- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
 - Time measured with infinitesimal precision.



The outcome space is not discrete

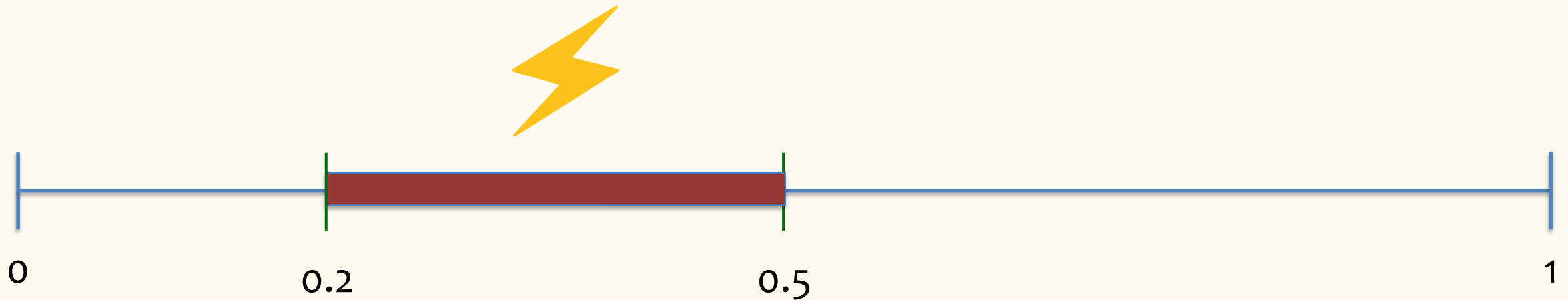
Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
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Lightning strikes a pole within a one-minute time frame

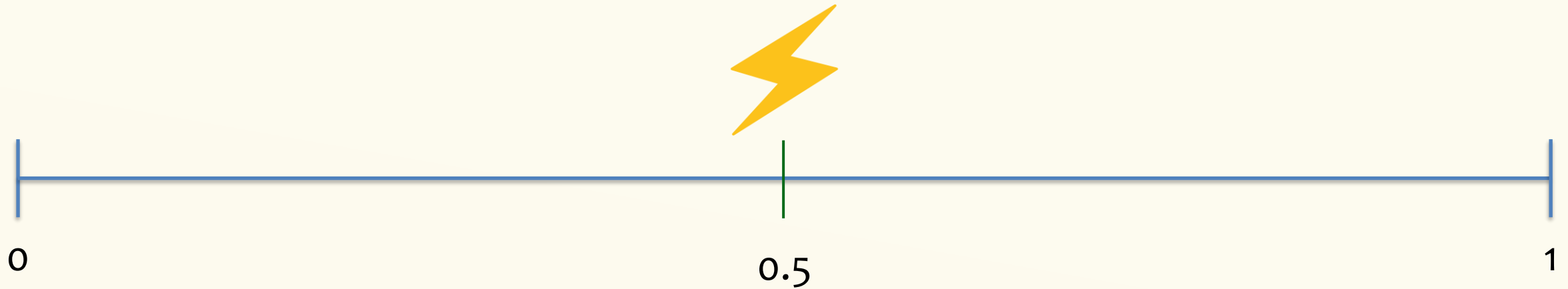
- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



$$\mathbb{P}(0.2 \leq T \leq 0.5) =$$

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely




$$\mathbb{P}(T = 0.5) =$$

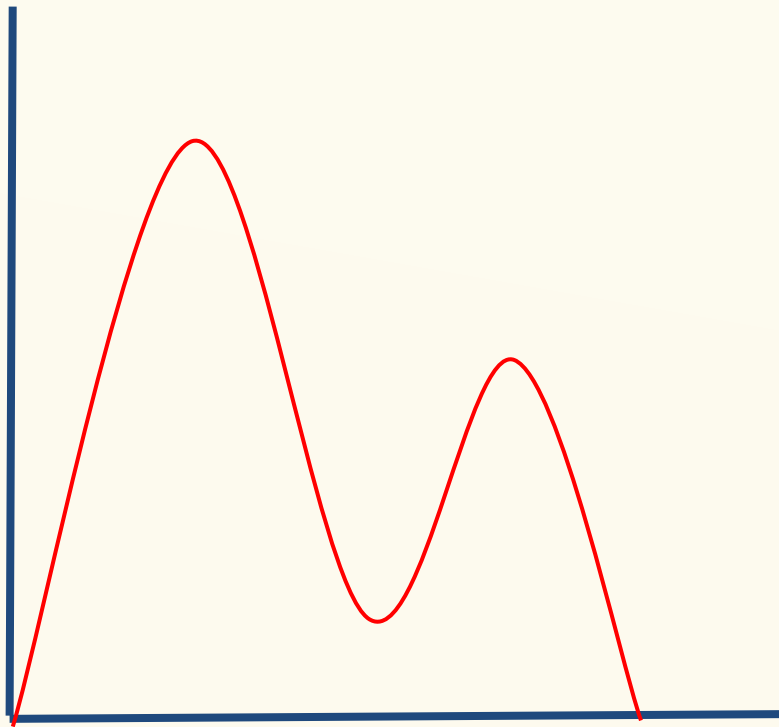
Bottom line

- This gives rise to a different type of random variable
- $\mathbb{P}(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
 - $\mathbb{P}(T \in [0,1]) = 1$
 - $\mathbb{P}(T \in [a, b]) = b - a$
 - ...
- How do we model the behavior of T ?

Agenda

- Poisson RV
- Continuous Random Variables
- **Probability Density Function** 
- Cumulative Distribution Function
- Expectation and Variance

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \rightarrow \mathbb{R}$, such that

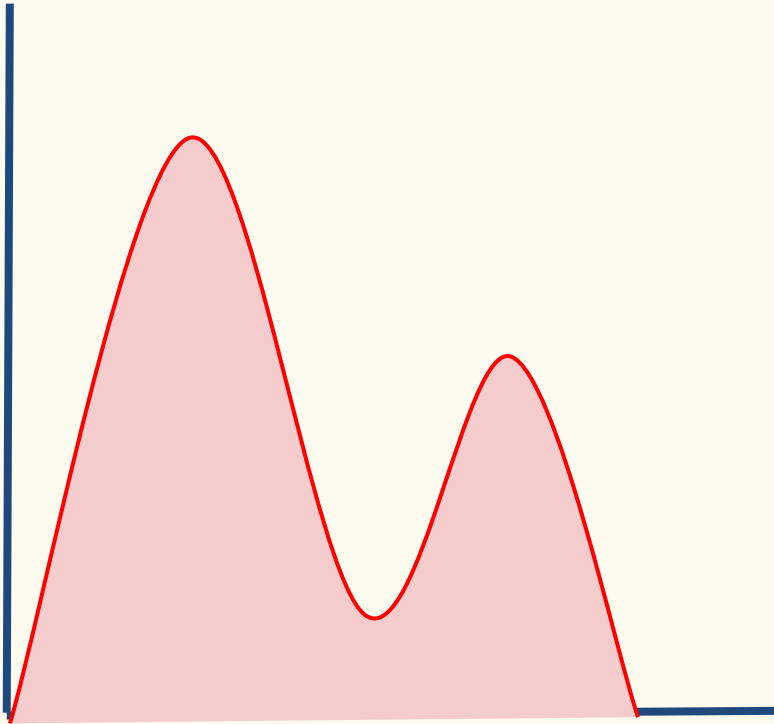


Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Probability Density Function - Intuition

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x) dx = 1$

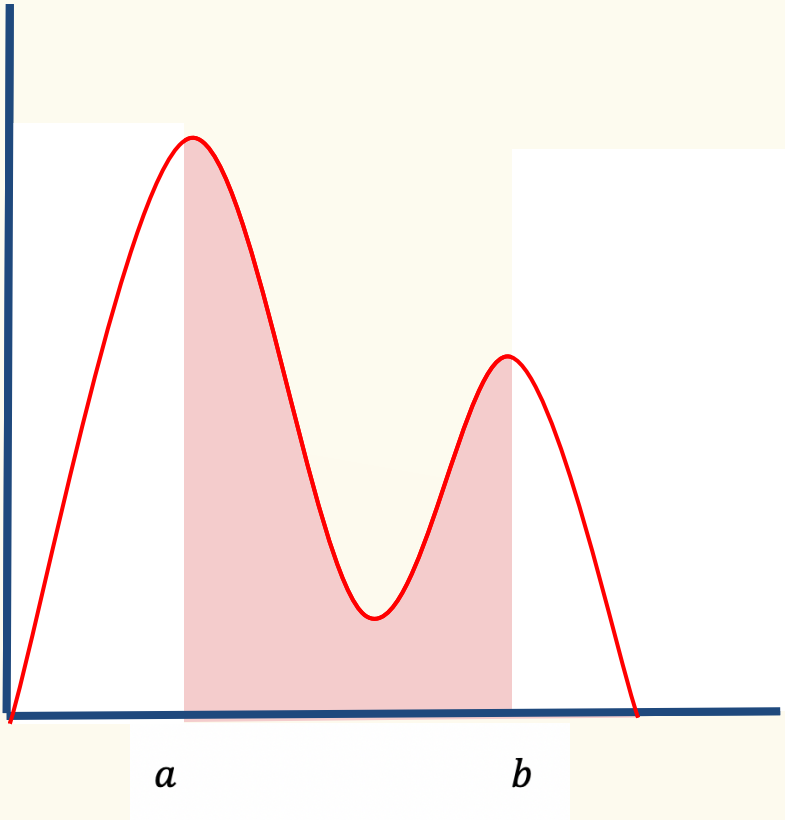


Probability Density Function - Intuition

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$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$



Probability Density Function - Intuition



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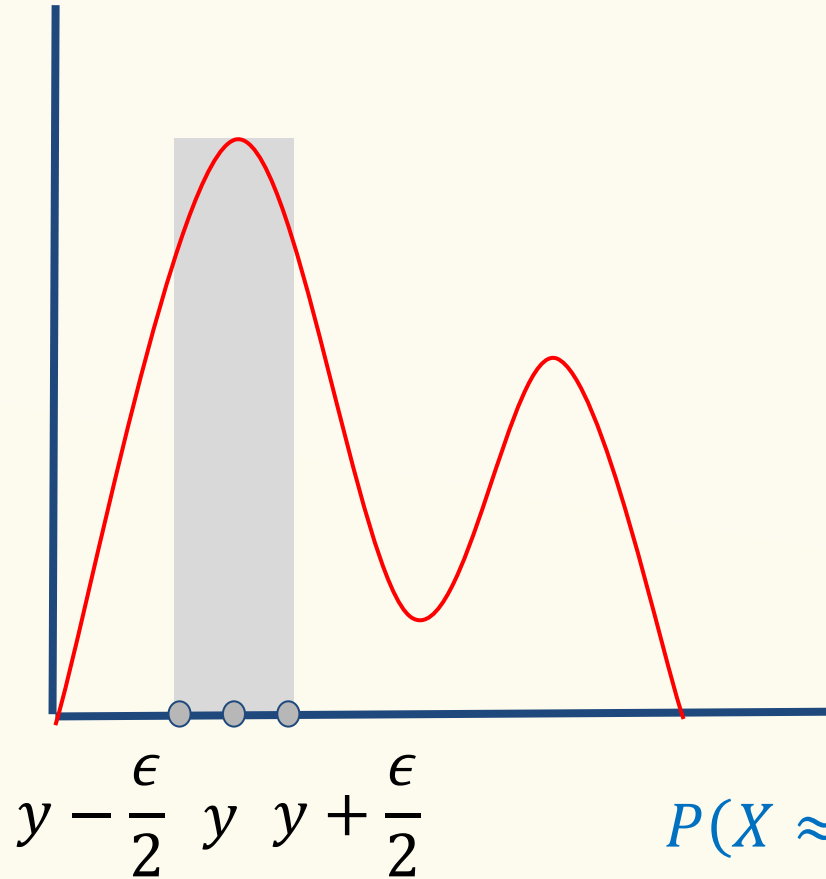
$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x) dx = 0$$



Density \neq Probability

$$f_X(y) \neq 0 \quad \mathbb{P}(X = y) = 0$$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

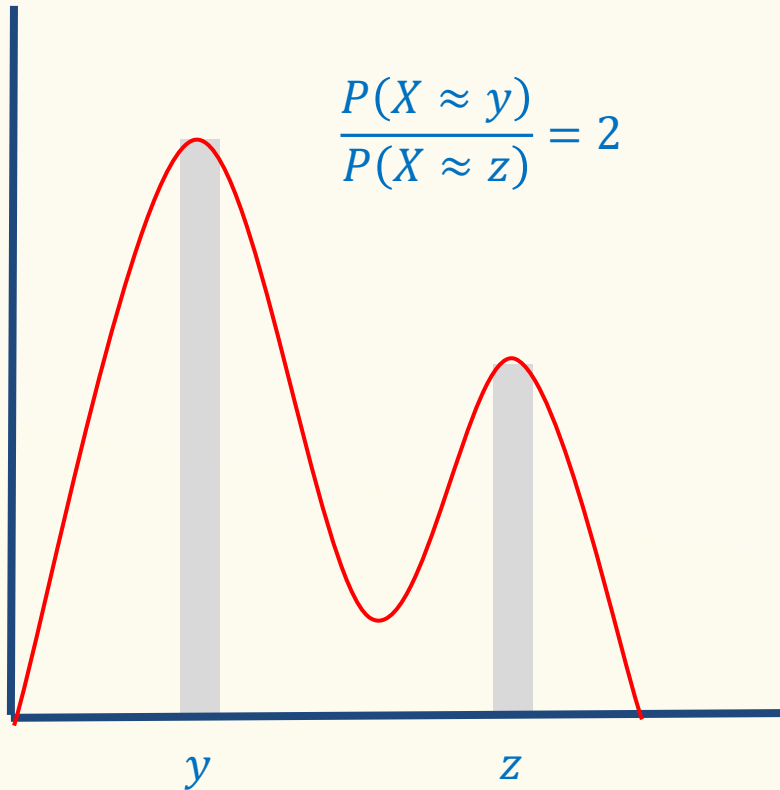
Normalization: $\int_{-\infty}^{+\infty} f_X(x) dx = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x) dx = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y - \frac{\epsilon}{2}}^{y + \frac{\epsilon}{2}} f_X(x) dx \approx \epsilon f_X(y)$$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x) dx = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

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$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \rightarrow \mathbb{R}$, such that

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x) dx = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x) dx = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y - \frac{\epsilon}{2}}^{y + \frac{\epsilon}{2}} f_X(x) dx \approx \epsilon f_X(y)$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

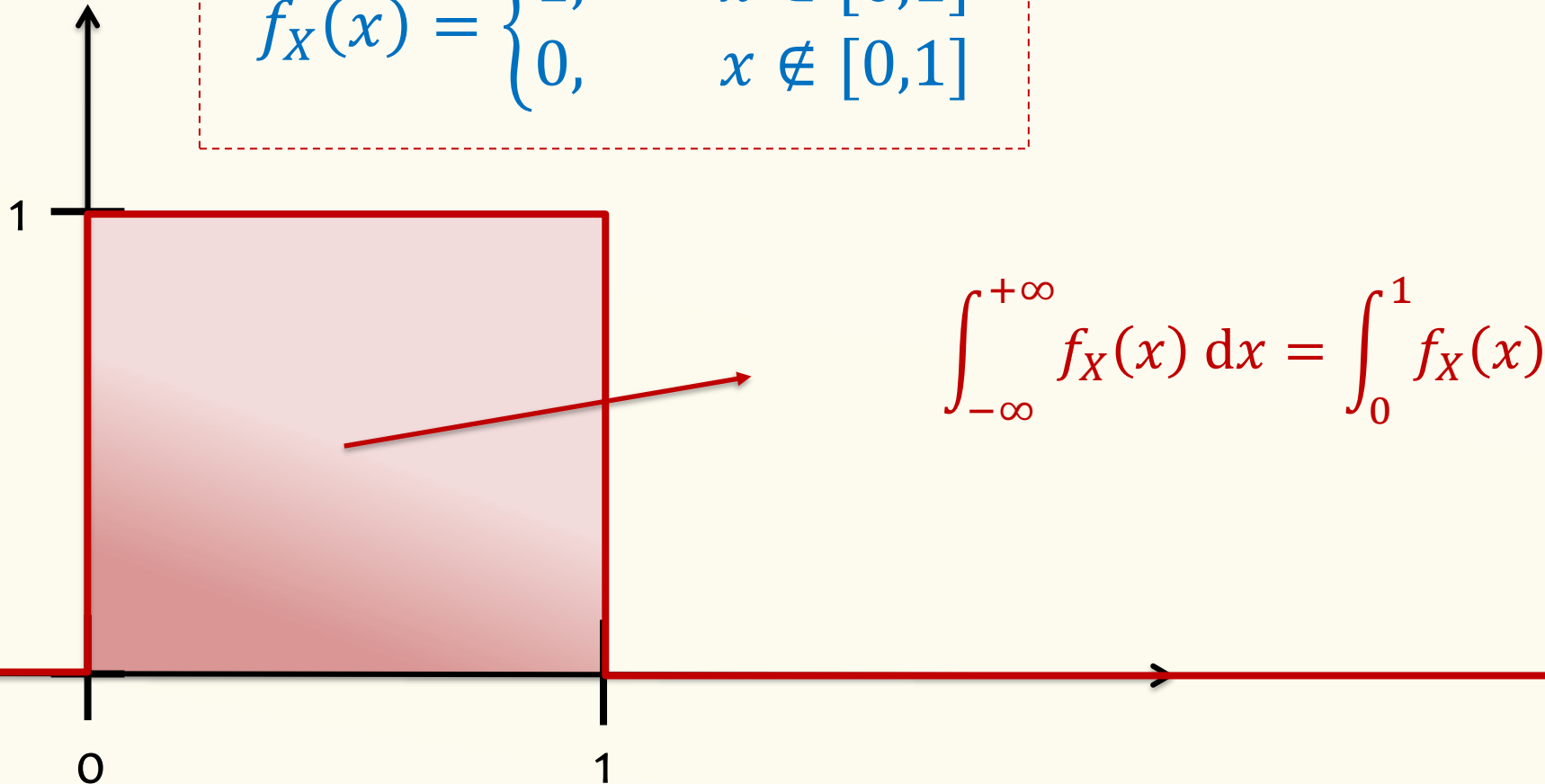
PDF of Uniform RV

$$X \sim \text{Unif}(0,1)$$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x) dx = 1$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x) dx = \int_0^1 f_X(x) dx = 1 \cdot 1 = 1$$

Probability of Event

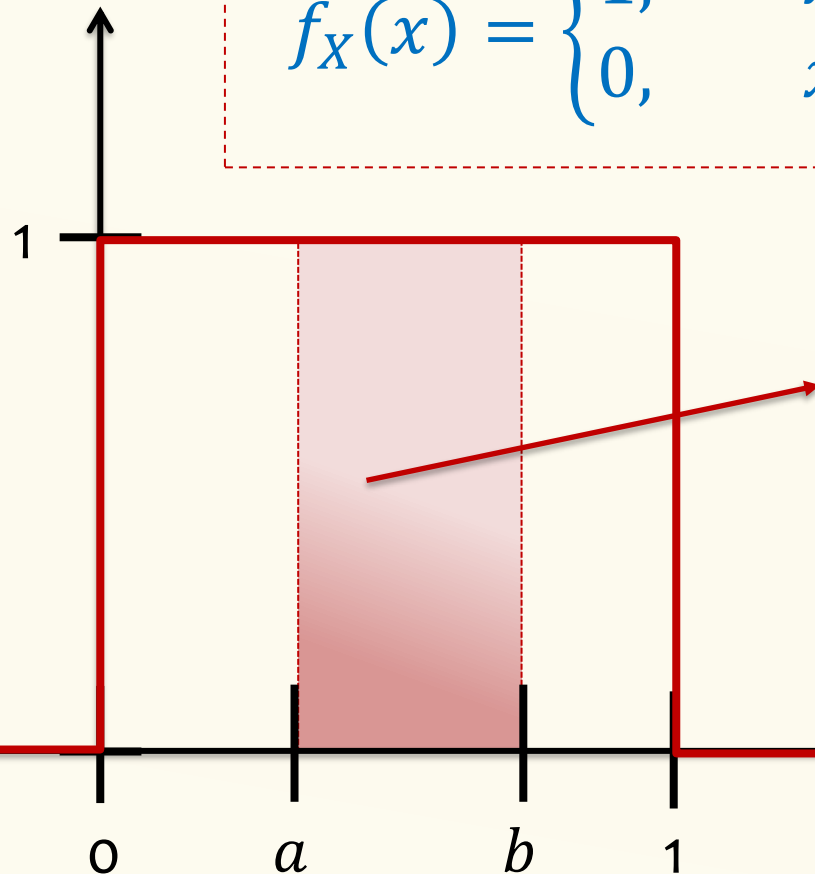
$X \sim \text{Unif}(0,1)$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

Normalization: $\int_{-\infty}^{+\infty} f_X(x) dx = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x) dx$$



1. If $0 \leq a$ and $b \leq 1$

$$\mathbb{P}(a \leq X \leq b) = b - a$$

2. If $a < 0$ and $0 \leq b \leq 1$

$$\mathbb{P}(a \leq X \leq b) = b$$

3. If $a \geq 0$ and $b > 1$

$$\mathbb{P}(a \leq X \leq b) = b - a$$

4. If $a < 0$ and $b > 1$

$$\mathbb{P}(a \leq X \leq b) = 1$$

- A. All of them are correct
- B. Only 1, 2, 4 are right
- C. Only 1 is right
- D. Only 1 and 2 are right**

PDF of Uniform RV

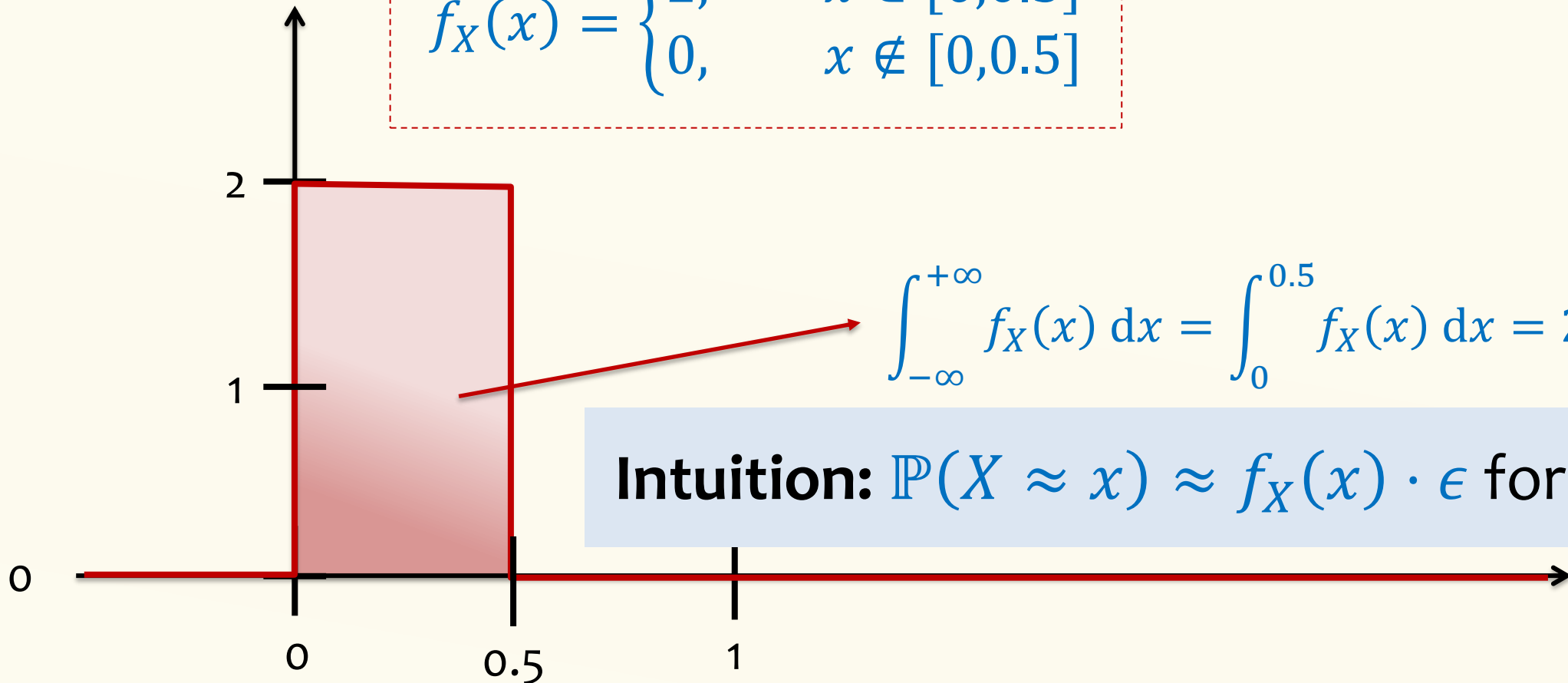
$$X \sim \text{Unif}(0,0.5)$$



Density \neq Probability

$f_X(x) \gg 1$ is possible!

$$f_X(x) = \begin{cases} 2, & x \in [0,0.5] \\ 0, & x \notin [0,0.5] \end{cases}$$

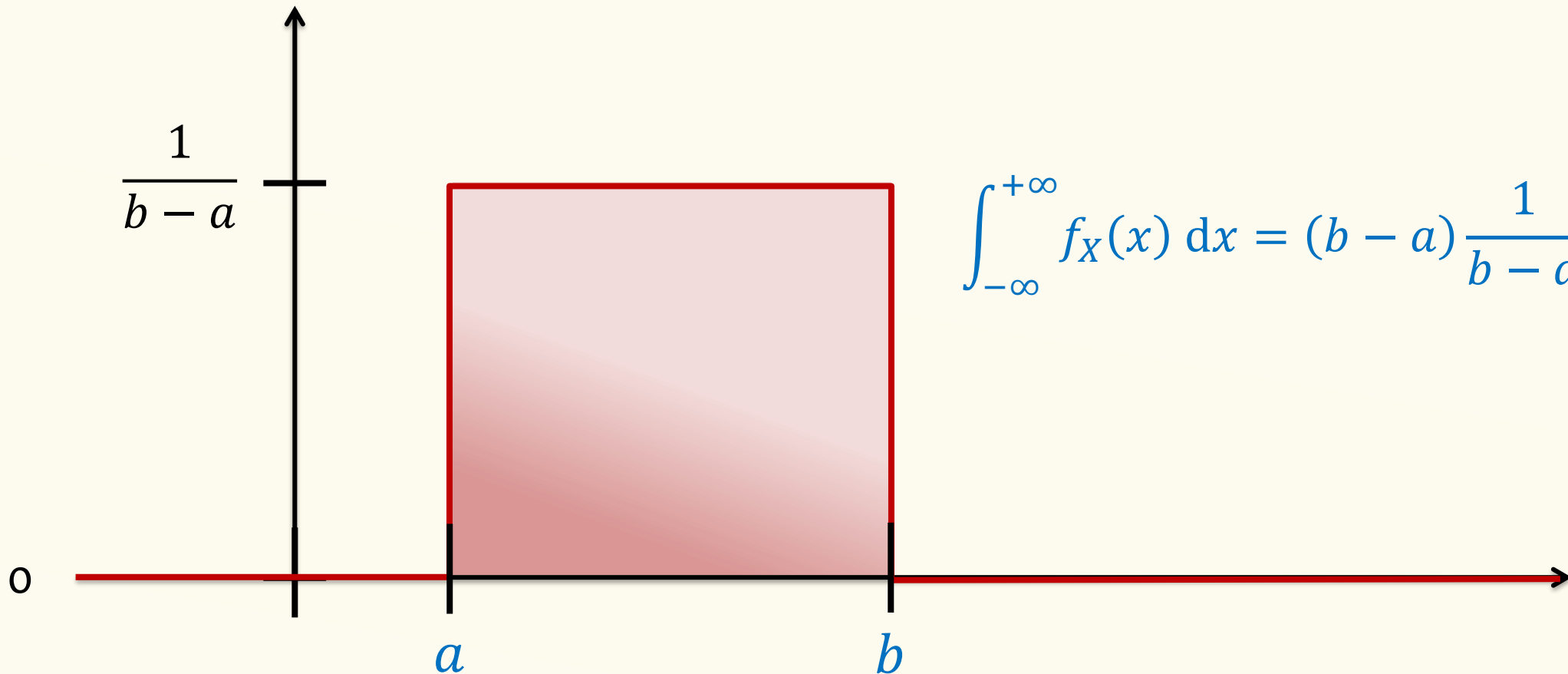


Intuition: $\mathbb{P}(X \approx x) \approx f_X(x) \cdot \epsilon$ for small ϵ

Uniform Distribution


$X \sim \text{Unif}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x) dx = (b-a) \frac{1}{b-a} = 1$$

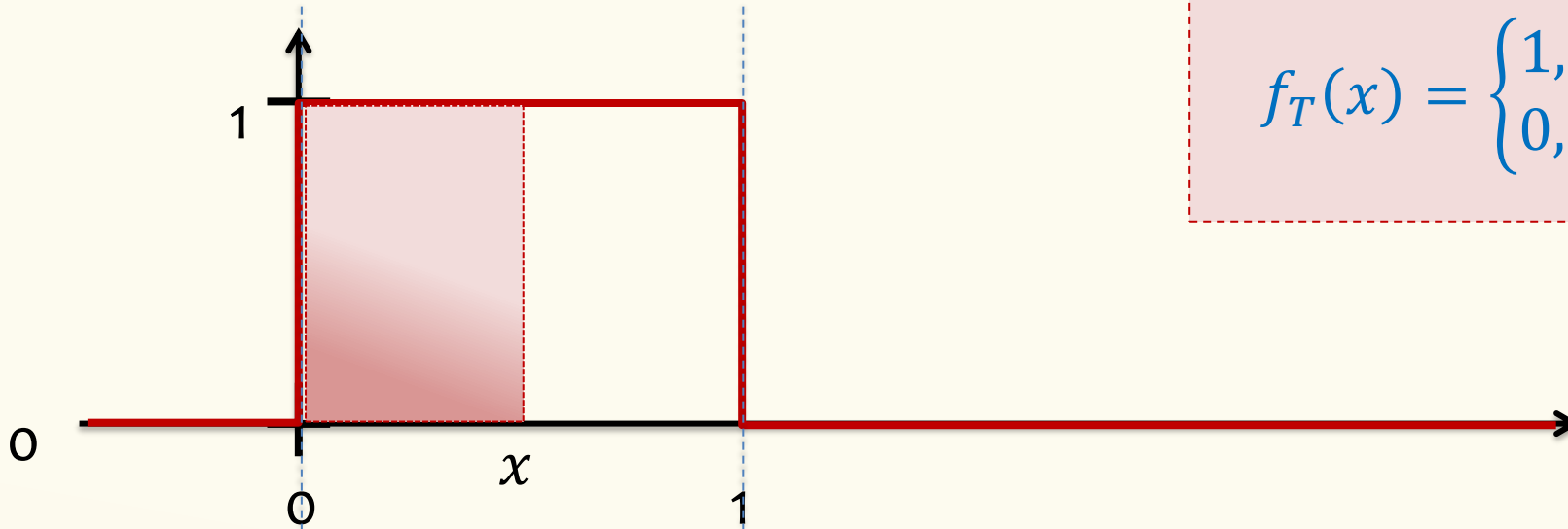
Agenda

- Poisson RV
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- Probability Density Function
- Cumulative Distribution Function 
- Expectation and Variance

Example. $T \sim \text{Unif}(0,1)$

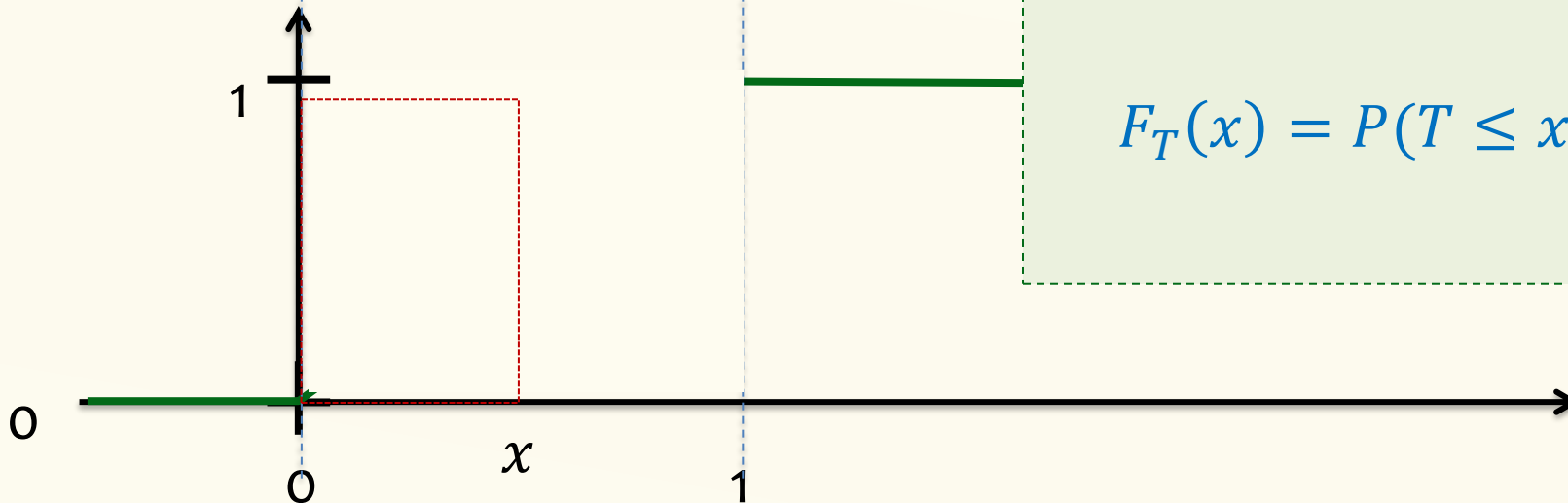
Probability Density Function

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



Cumulative Distribution Function

$$F_T(x) = P(T \leq x) = \begin{cases} 0 & x \leq 0 \\ ? & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases}$$



Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = \mathbb{P}(X \leq a) = \int_{-\infty}^a f_X(x) dx$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F(x)$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = \mathbb{P}(X \leq a) = \int_{-\infty}^a f_X(x) dx$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F(x)$

Therefore: $\mathbb{P}(X \in [a, b]) = F(b) - F(a)$

F_X is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

$\lim_{a \rightarrow -\infty} F_X(a) = P(X \leq -\infty) = 0$ $\lim_{a \rightarrow +\infty} F_X(a) = P(X \leq +\infty) = 1$

From Discrete to Continuous

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t < x} p_X(t)$	$F_X(x) = \int_{-\infty}^x f_X(t) dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x) dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_x g(x)p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x)f_X(x) dx$

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Expectation of a Continuous RV

Definition. The **expected value** of a continuous RV X is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

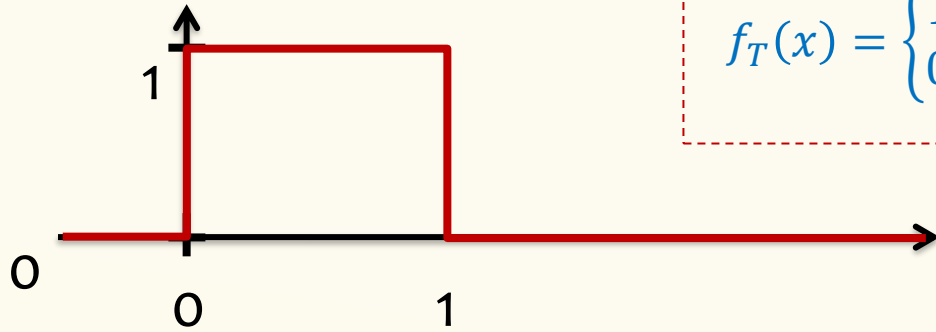
Fact. $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

Definition. The **variance** of a continuous RV X is defined as

$$\text{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}(X))^2 \, dx = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

Expectation of a Continuous RV

Example. $T \sim \text{Unif}(0,1)$



$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

Definition.

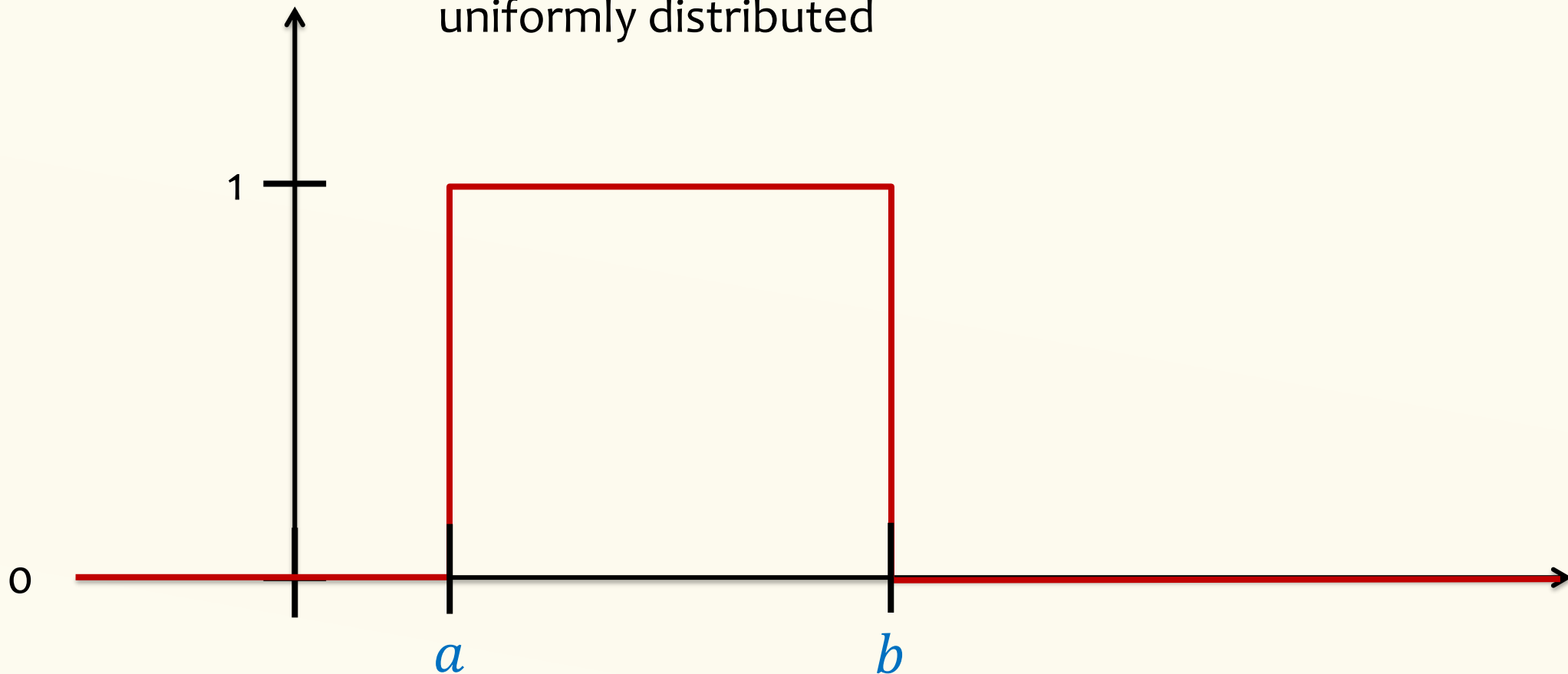
$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

Uniform Distribution

$$X \sim \text{Unif}(a, b)$$

We also say that X follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



Uniform Density – Expectation

$$X \sim \text{Unif}(a, b)$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

Uniform Density – Expectation

$$X \sim \text{Unif}(a, b)$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx$$

$$\begin{aligned} &= \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \left(\frac{x^2}{2} \right) \Big|_a^b = \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right) \\ &= \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2} \end{aligned}$$

Uniform Density – Variance

$$X \sim \text{Unif}(a, b)$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, dx$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

Uniform Density – Variance

$$X \sim \text{Unif}(a, b)$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, dx$$

$$= \frac{1}{b-a} \int_a^b x^2 \, dx = \frac{1}{b-a} \left(\frac{x^3}{3} \right) \Big|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

Uniform Density – Variance

$$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3} \quad \mathbb{E}(X) = \frac{a + b}{2}$$

$$X \sim \text{Unif}(a, b)$$

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b - a)^2}{12}$$