CSE 312

Foundations of Computing II

Lecture 12: Zoo of Discrete RVs

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Motivation: “Named” Random Variables

Random Variables that show up all over the place.

– Easily solve a problem by recognizing it’s a special case of one of these random variables.

Each RV introduced today will show:

– A general situation it models

– Its name and parameters

– Its PMF, Expectation, and Variance

Welcome to the Zoo! (Preview)

$X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$

$$E[X] = \frac{a + b}{2}$$

$$Var(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$E[X] = p$$

$$Var(X) = p(1 - p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np$$

$$Var(X) = np(1 - p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$

$$E[X] = \frac{1}{p}$$

$$Var(X) = \frac{1 - p}{p^2}$$

$X \sim \text{Poisson}(\lambda)$

$$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$E[X] = \lambda$$

$$Var(X) = \lambda$$

+ bonus ones!

Agenda

- Discrete Uniform Random Variables
- Bernoulli Random Variables
- Binomial Random Variables
- Geometric Random Variables
  - Bonus material
- Poisson

Discrete Uniform Random Variables

A discrete random variable X **equally likely** to take any (int.) value between integers a and b (inclusive), is uniform.

Notation: $X \sim Unif(a, b)$

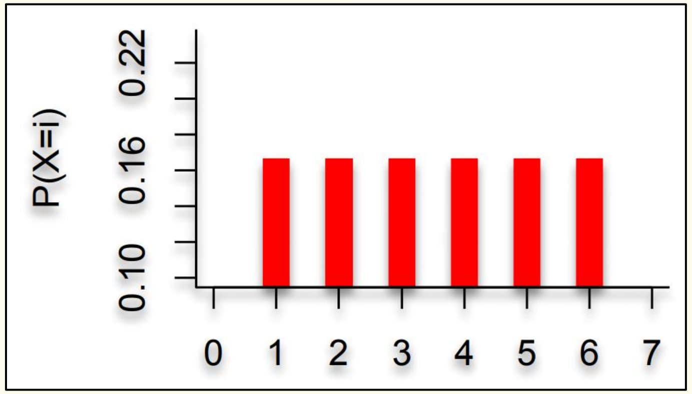
PMF:

Expectation:

Variance:

Example: value shown on one roll of a fair die

$X \sim Unif(1, 6)$



Discrete Uniform Random Variables

A discrete random variable X **equally likely** to take any (int.) value between integers a and b (inclusive), is uniform.

Notation: $X \sim \mathrm{Unif}(a, b)$

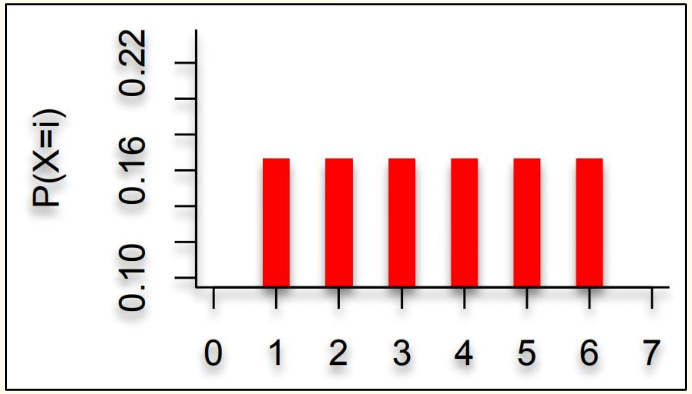
PMF: $\Pr(X = i) = \frac{1}{b - a + 1}$

Expectation: $\mathrm{E}[X] = \frac{a+b}{2}$

Variance: $\mathrm{Var}(X) = \frac{(b-a)(b-a+2)}{12}$

Example: value shown on one roll of a fair die is $\mathrm{Unif}(1,6)$:

- $\Pr(X = i) = 1/6$
- $E[X] = 7/2$
- $\mathrm{Var}(X) = 35/12$



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Indicator?

Bernoulli Random Variables

X_1 X_2 X_3 X_i = i^{th} coin flip

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise is called a Bernoulli random variable.

Notation: $X \sim \text{Ber}(p)$

PMF: $\Pr(X = 1) = p, \Pr(X = 0) = 1 - p$

Expectation: $E[X] = p$

Variance:

Bernoulli Random Variables

A random variable X that takes value 1 ("Success") with probability p, and 0 ("Failure") otherwise. X is called a Bernoulli random variable.

Notation: $X \sim \mathrm{Ber}(p)$

PMF: $\Pr(X = 1) = p, \Pr(X = 0) = 1 - p$

Expectation: $\mathrm{E}[X] = p$ Note: $\mathrm{E}[X^2] = p$

Variance: $\mathrm{Var}(X) = E[X^2] - E[X]^2 = p - p^2 = p(1 - p)$

Examples:

- Coin flip
- Randomly guessing on a MC test question
- A server in a cluster fails

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Binomial Random Variables

A discrete random variable X that is the number of successes in n independent random variables $Y_i \sim \mathrm{Ber}(p)$ is a Binomial random variable where $X = \sum_{i=1}^{n} Y_i$

Examples:

- # of heads in n coin flips
- # of 1s in a randomly generated n bit string
- # of servers that fail in a cluster of n computers
- # of bit errors in file written to disk
- # of elements in a bucket of a large hash table

Poll:

$\Pr(X = k)$ =

a. $p^k(1-p)^{n-k}$

b. np

c. $\binom{n}{k}p^k(1-p)^{n-k}$

d. $\binom{n}{n-k}p^k(1-p)^{n-k}$

Binomial Random Variables

A discrete random variable X that is the number of successes in n independent random variables $Y_i \sim \mathrm{Ber}(p)$. X is a Binomial random variable where $X = \sum_{i=1}^{n} Y_i$

Notation: $X \sim \mathrm{Bin}(n, p)$

PMF: $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation:

Variance:

Poll:

	Mean	Variance
a.	p	p
b.	np	$\mathrm{np}(1-p)$
c.	np	np^2
d.	np	n^2p

Binomial Random Variables

A discrete random variable X that is the number of successes in n independent random variables $Y_i \sim \mathrm{Ber}(p)$. X is a Binomial random variable where $X = \sum_{i=1}^{n} Y_i$

Notation: $X \sim \mathrm{Bin}(n, p)$

PMF: $\Pr(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

Expectation: $\mathrm{E}[X] = np$

Variance: $\mathrm{Var}(X) = np(1-p)$

Mean, Variance of the Binomial

If $Y_1, Y_2, \ldots, Y_n \sim \mathrm{Ber}(p)$ and independent (i.i.d), then
$X = \sum_{i=1}^{n} Y_i, \quad X \sim \mathrm{Bin}(n, p)$

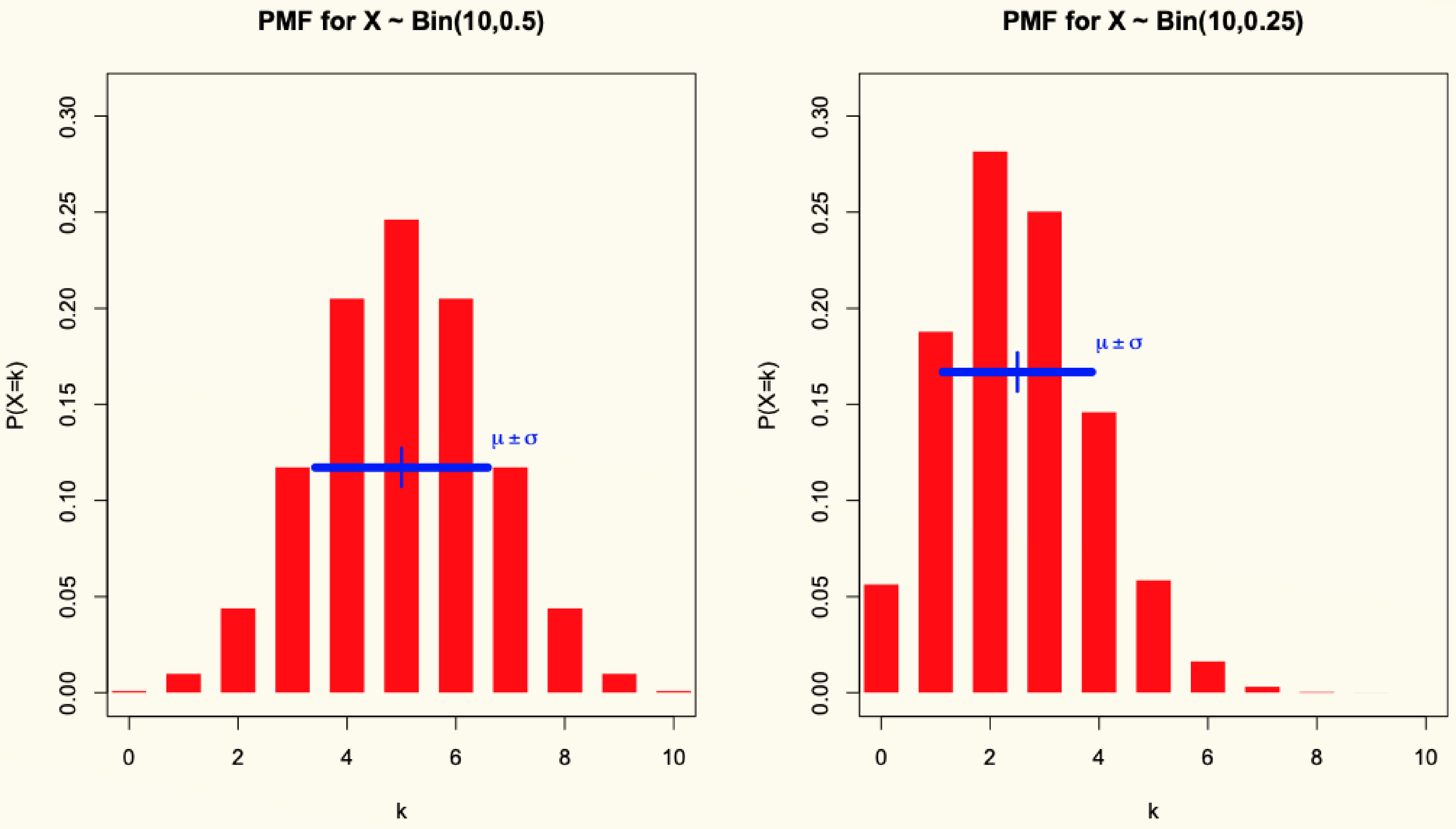
Claim $E[X] = np$

$$E[X] = E\left[\sum_{i=1}^{n} Y_i\right] = \sum_{i=1}^{n} E[Y_i] = nE[Y_1] = np$$

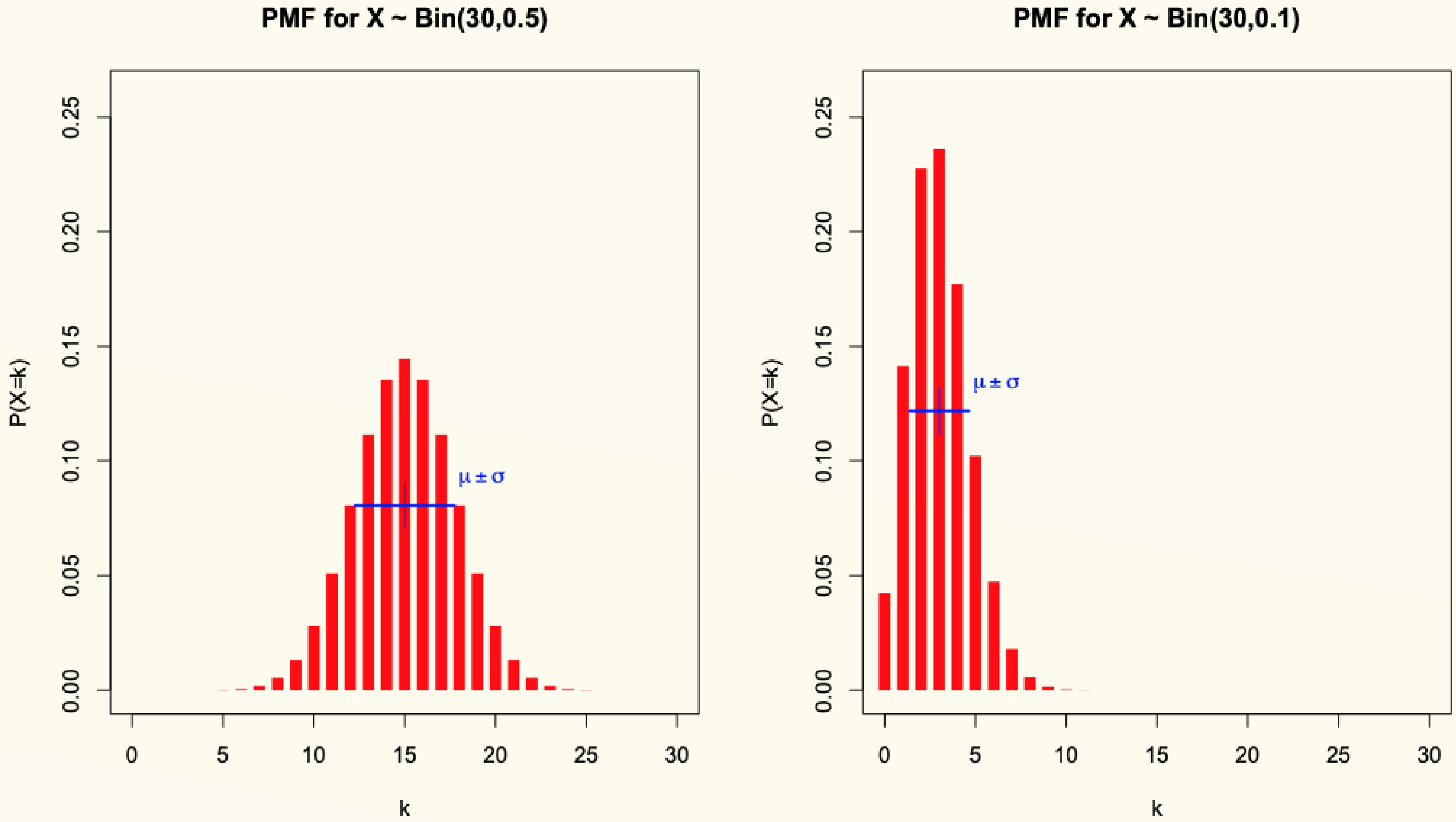
Claim $Var(X) = np(1-p)$

$$Var(X) = Var\left(\sum_{i=1}^{n} Y_i\right) = \sum_{i=1}^{n} Var(Y_i) = nVar(Y_1) = np(1-p)$$

Binomial PMFs



Binomial PMFs



Example

Sending a binary message of length 1024 bits over a network with probability 0.999 of correctly sending each bit in the message without corruption (independent of other bits). Let X be the number of corrupted bits. What is $\mathrm{E}[X]$?

Poll:

a. 1022.99
b. 1.024
c. 1.02298
d. 1
e. Not enough information to compute

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Geometric Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim \text{Ber}(p)$ before seeing the first success. X is called a Geometric random variable with parameter p.

Notation: $X \sim \text{Geo}(p)$

PMF:

Expectation:

Variance:

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

Geometric Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim \mathrm{Ber}(p)$ before seeing the first success. X is called a Geometric random variable with parameter p.

Notation: $X \sim \mathrm{Geo}(p)$

PMF: $\Pr(X = k) = (1-p)^{k-1}p$

Expectation: $\mathrm{E}[X] = \frac{1}{p}$

Variance: $\mathrm{Var}(X) = \frac{1-p}{p^2}$

Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

Example: Music Lessons

Your music teacher requires you to play a 1000 note song without mistake. You have been practicing, so you have a probability of 0.999 of getting each note correct (independent of the others). If you mess up a single note in the song, you must start over and play from the beginning. Let X be the number of times you have to play the song from the start. What is $\mathrm{E}[X]$?

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Bonus: Negative Binomial Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim Ber(p)$ before seeing the r^{th} success. Equivalently, $X = \sum_{i=1}^{r} Z_i$ where $\mathrm{Z}_i \sim \mathrm{Geo}(p)$. X is called a Negative Binomial random variable with parameters r, p.

Notation: $X \sim \mathrm{NegBin}(r, p)$

PMF:

Expectation:

Variance:

Bonus: Negative Binomial Random Variables

A discrete random variable X that models the number of independent trials $Y_i \sim Ber(p)$ before seeing the r^{th} success. Equivalently, $X = \sum_{i=1}^{r} Z_i$ where $\mathrm{Z}_i \sim \mathrm{Geo}(p)$. X is called a Negative Binomial random variable with parameters r, p.

Notation: $X \sim \mathrm{NegBin}(r, p)$

PMF: $\Pr(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

Expectation: $\mathrm{E}[X] = \frac{r}{p}$

Variance: $\mathrm{Var}(X) = \frac{r(1-p)}{p^2}$

Bonus: Hypergeometric Random Variables

A discrete random variable X that measures the number of white balls you draw when you draw n balls uniformly at random from a total of N of which K are white and the rest are black. X is called a Hypergeometric RV with parameters N, K, n.

Notation: $X \sim \text{HypGeo}(N, K, n)$

PMF:

Expectation:

Bonus: Hypergeometric Random Variables

A discrete random variable X that measures the number of white balls you draw when you draw n balls uniformly at random from a total of N of which K are white and the rest are black. X is called a Hypergeometric RV with parameters N, K, n.

Notation: $X \sim \text{HypGeo}(N, K, n)$

PMF: $\Pr(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$

Expectation: $\text{E}[X] = n\frac{K}{N}$

Variance: $\text{Var}(X) = n\frac{K(N-K)(N-n)}{N^2(N-1)}$

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Poisson Distribution

Siméon Denis Poisson
1781-1840

- Suppose "events" happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. *with parameter* λ (denoted X ~ Poi(λ)) and has distribution (PMF):

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

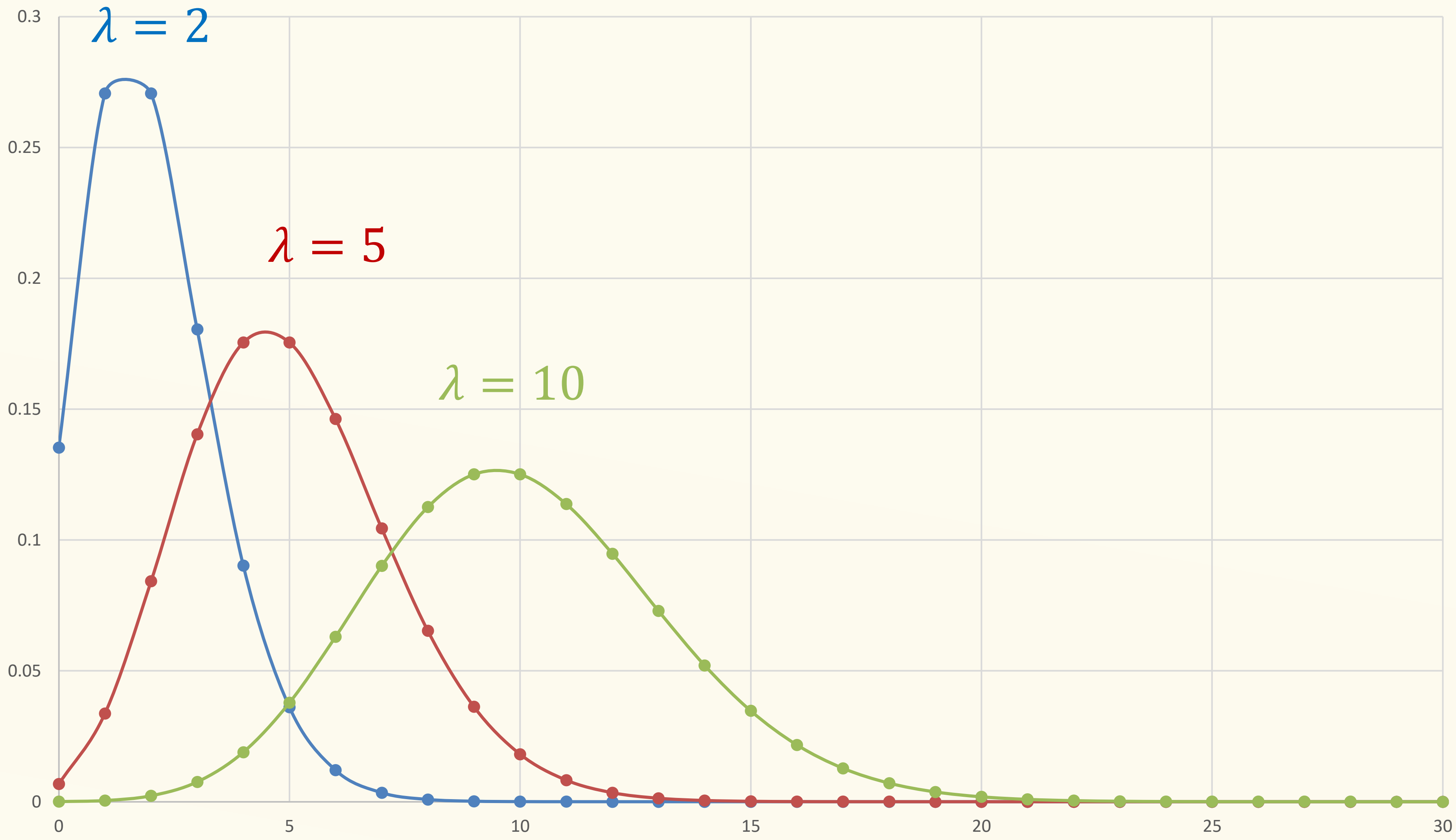
Several examples of "Poisson processes":

- # of cars passing through a certain town in 1 hour
- # of requests to web servers in a minute
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume fixed average rate

Probability Mass Function

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) =$$

Fact. $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$

Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}} = e^{-\lambda} e^{\lambda} = 1$$

Fact. $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$

Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then

$$\mathbb{E}(X) = \lambda$$

Proof. $\mathbb{E}(X) = \sum_{i=0}^{\infty} i \cdot \mathbb{P}(X = i)$

Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then

$$\mathbb{E}(X) = \lambda$$

Proof.

$$\mathbb{E}(X) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

= 1 (see prior slides!)

$$= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda$$

Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then $\mathrm{Var}(X) = \lambda$

Proof. $$\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 \quad = \lambda^2 + \lambda$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then $\mathrm{Var}(X) = \lambda$

Proof. $\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1)$$

$$= \lambda \left[\underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}(X) = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda$$

Similar to the previous proof
Verify offline.

$$\Rightarrow \quad \mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Notation: $X \sim \mathrm{Poi}(\lambda)$

PMF: $\Pr(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

Expectation: $\mathrm{E}[X] = \lambda$

Variance: $\mathrm{Var}(X) = \lambda$

Sum of Independent Poisson RVs

Theorem. Let $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $\mathrm{Z} = (X + Y)$. For all $k = 0,1,2,3 \ldots,$

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

More generally, let $X_1 \sim Poi(\lambda_1), \cdots, X_n \sim Poi(\lambda_n)$ such that $\lambda = \Sigma_i \lambda_i$.
Let $\mathrm{Z} = \Sigma_i X_i$

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

Poisson Example

There are two ERs in a small town that act independently. The first has an average of 4 patients admitted per hour, and the second has an average of 3. What is the likelihood that in the next hour, 10 patients are admitted across both ERs?