

CSE 312

# Foundations of Computing II

## Lecture 12: Zoo of Discrete RVs



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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊

# Motivation: “Named” Random Variables

Random Variables that show up all over the place.

- Easily solve a problem by recognizing it’s a special case of one of these random variables.

Each RV introduced today will show:

- A general situation it models
- Its name and parameters
- Its PMF, Expectation, and Variance

# Welcome to the Zoo! (Preview)



$X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$

$$E[X] = \frac{a + b}{2}$$

$$\text{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$E[X] = p$$

$$\text{Var}(X) = p(1 - p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np$$

$$\text{Var}(X) = np(1 - p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$

$$E[X] = \frac{1}{p}$$

$$\text{Var}(X) = \frac{1 - p}{p^2}$$

$X \sim \text{Poisson}(\lambda)$

$$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$E[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

+ bonus ones!

# Agenda

- Discrete Uniform Random Variables ◀
- Bernoulli Random Variables
- Binomial Random Variables
- Geometric Random Variables
  - Bonus material
- Poisson

# Discrete Uniform Random Variables

A discrete random variable  $X$  **equally likely** to take any (int.) value between integers  $a$  and  $b$  (inclusive), is **uniform**.

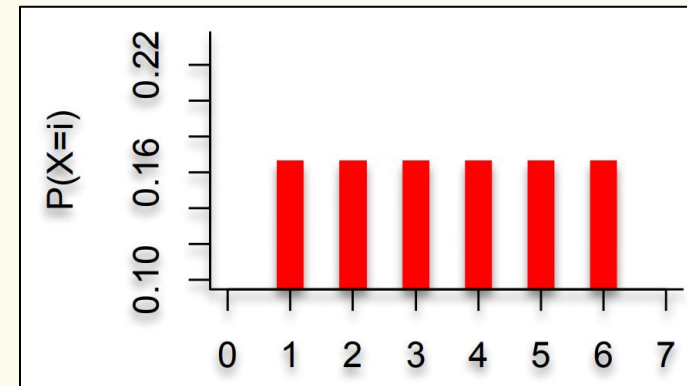
**Notation:**

**PMF:**

**Expectation:**

**Variance:**

**Example:** value shown on one roll of a fair die



# Discrete Uniform Random Variables

A discrete random variable  $X$  **equally likely** to take any (int.) value between integers  $a$  and  $b$  (inclusive), is **uniform**.

**Notation:**  $X \sim \text{Unif}(a, b)$

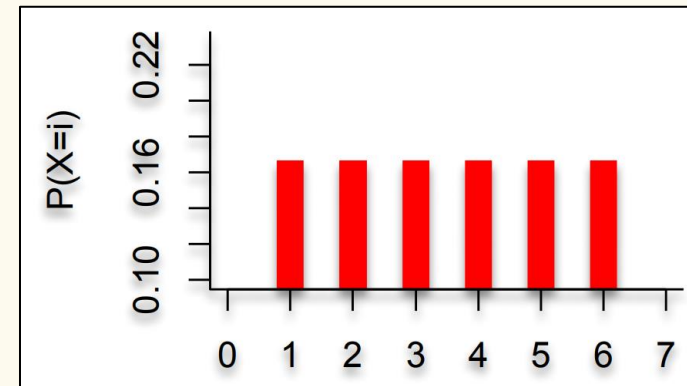
**PMF:**  $\Pr(X = i) = \frac{1}{b - a + 1}$

**Expectation:**  $E[X] = \frac{a+b}{2}$


**Variance:**  $\text{Var}(X) = \frac{(b-a)(b-a+1)}{12}$

**Example:** value shown on one roll of a fair die is  $\text{Unif}(1,6)$ :

- $\Pr(X = i) = 1/6$
- $E[X] = 7/2$
- $\text{Var}(X) = 35/12$



# Agenda

- Discrete Uniform Random Variables
- **Bernoulli Random Variables** 
- Binomial Random Variables
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# Bernoulli Random Variables

A random variable  $X$  that takes value **1** (“Success”) with probability  $p$ , and **0** (“Failure”) otherwise is called a **Bernoulli random variable**.

**Notation:**  $X \sim \text{Ber}(p)$

**PMF:**  $\Pr(X = 1) = p, \Pr(X = 0) = 1 - p$

**Expectation:**

**Variance:**



# Bernoulli Random Variables

A random variable  $X$  that takes value **1** (“Success”) with probability  $p$ , and **0** (“Failure”) otherwise.  $X$  is called a **Bernoulli random variable**.

**Notation:**  $X \sim \text{Ber}(p)$

**PMF:**  $\Pr(X = 1) = p, \Pr(X = 0) = 1 - p$


**Expectation:**  $E[X] = p$       Note:  $E[X^2] = p$

**Variance:**  $\text{Var}(X) = E[X^2] - E[X]^2 = p - p^2 = p(1 - p)$

## Examples:

- Coin flip
- Randomly guessing on a MC test question
- A server in a cluster fails

# Agenda

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# Binomial Random Variables

A discrete random variable  $X$  that is the number of successes in  $n$  independent random variables  $Y_i \sim \text{Ber}(p)$  is a **Binomial random variable** where  $X = \sum_{i=1}^n Y_i$

## Examples:

- # of heads in  $n$  coin flips
- # of 1s in a randomly generated  $n$  bit string
- # of servers that fail in a cluster of  $n$  computers
- # of bit errors in file written to disk
- # of elements in a bucket of a large hash table

**Poll:**

$\Pr(X = k) =$

a.  $p^k(1-p)^{n-k}$

b.  $np$

c.  $\binom{n}{k}p^k(1-p)^{n-k}$

d.  $\binom{n}{n-k}p^k(1-p)^{n-k}$

# Binomial Random Variables

A discrete random variable  $X$  that is the number of successes in  $n$  independent random variables  $Y_i \sim \text{Ber}(p)$ .  $X$  is a **Binomial random variable** where  $X = \sum_{i=1}^n Y_i$

**Notation:**  $X \sim \text{Bin}(n, p)$

**PMF:**  $\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$

**Expectation:**

**Variance:**

**Poll:**

	Mean	Variance
a.	$p$	$p$
b.	$np$	$np(1 - p)$
c.	$np$	$np^2$
d.	$np$	$n^2p$

# Binomial Random Variables

A discrete random variable  $X$  that is the number of successes in  $n$  independent random variables  $Y_i \sim \text{Ber}(p)$ .  $X$  is a **Binomial random variable** where  $X = \sum_{i=1}^n Y_i$

**Notation:**  $X \sim \text{Bin}(n, p)$

**PMF:**  $\Pr(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$

**Expectation:**  $E[X] = np$

**Variance:**  $\text{Var}(X) = np(1 - p)$

# Mean, Variance of the Binomial

If  $Y_1, Y_2, \dots, Y_n \sim \text{Ber}(p)$  and independent (i.i.d), then

$$X = \sum_{i=1}^n Y_i, \quad X \sim \text{Bin}(n, p)$$

Claim  $E[X] = np$

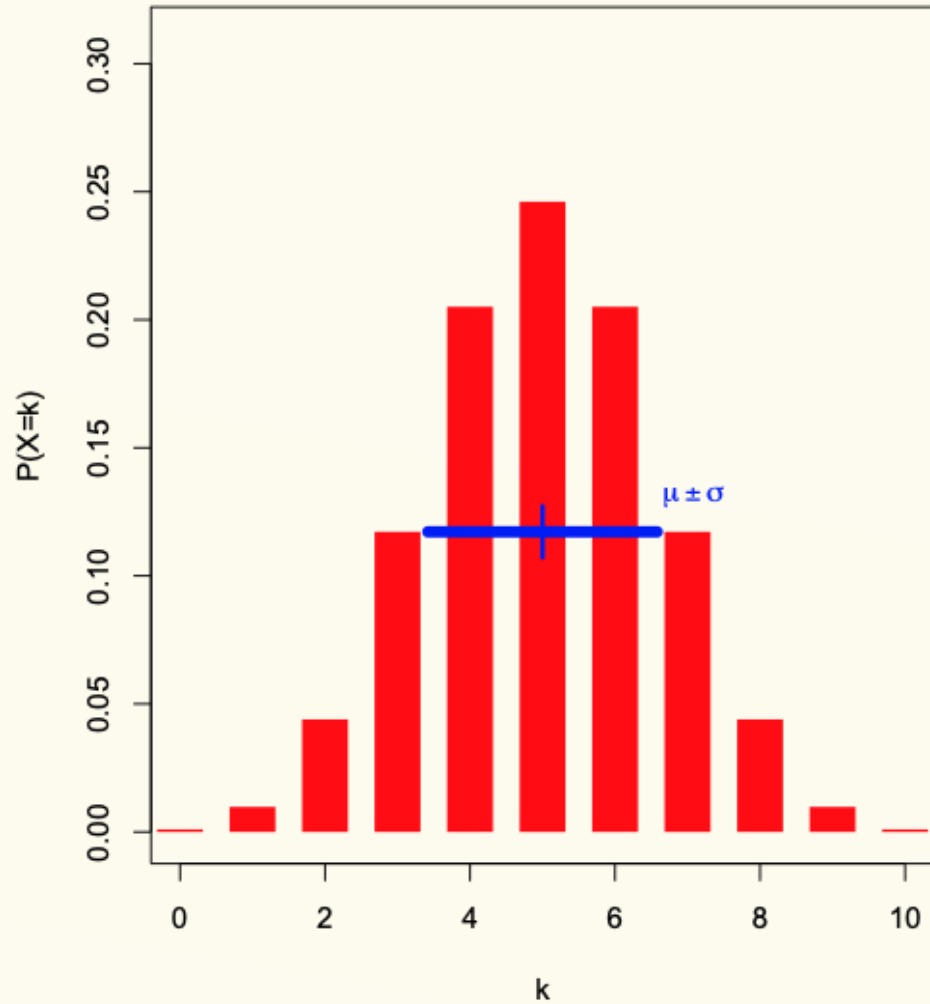
$$E[X] = E\left[\sum_{i=1}^n Y_i\right] = \sum_{i=1}^n E[Y_i] = nE[Y_1] = np$$

Claim  $\text{Var}(X) = np(1 - p)$

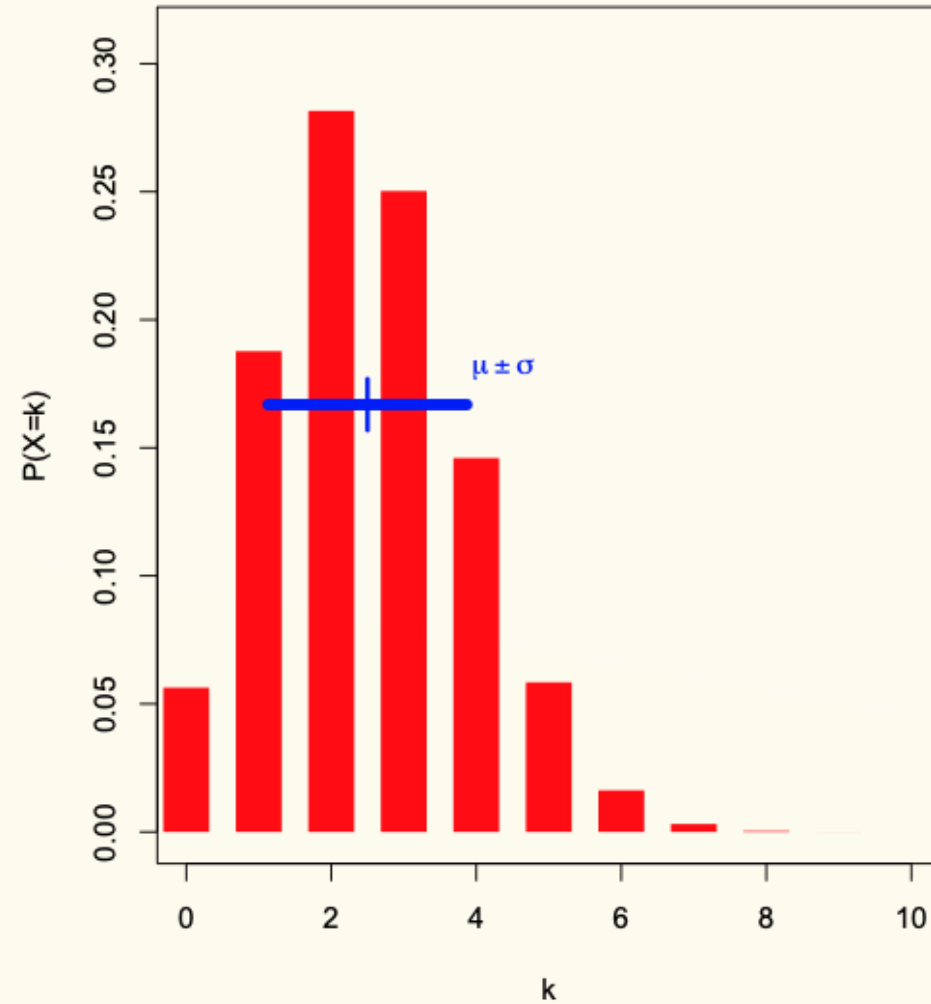
$$\text{Var}(X) = \text{Var}\left(\sum_{i=1}^n Y_i\right) = \sum_{i=1}^n \text{Var}(Y_i) = n\text{Var}(Y_1) = np(1 - p)$$

# Binomial PMFs

PMF for  $X \sim \text{Bin}(10,0.5)$

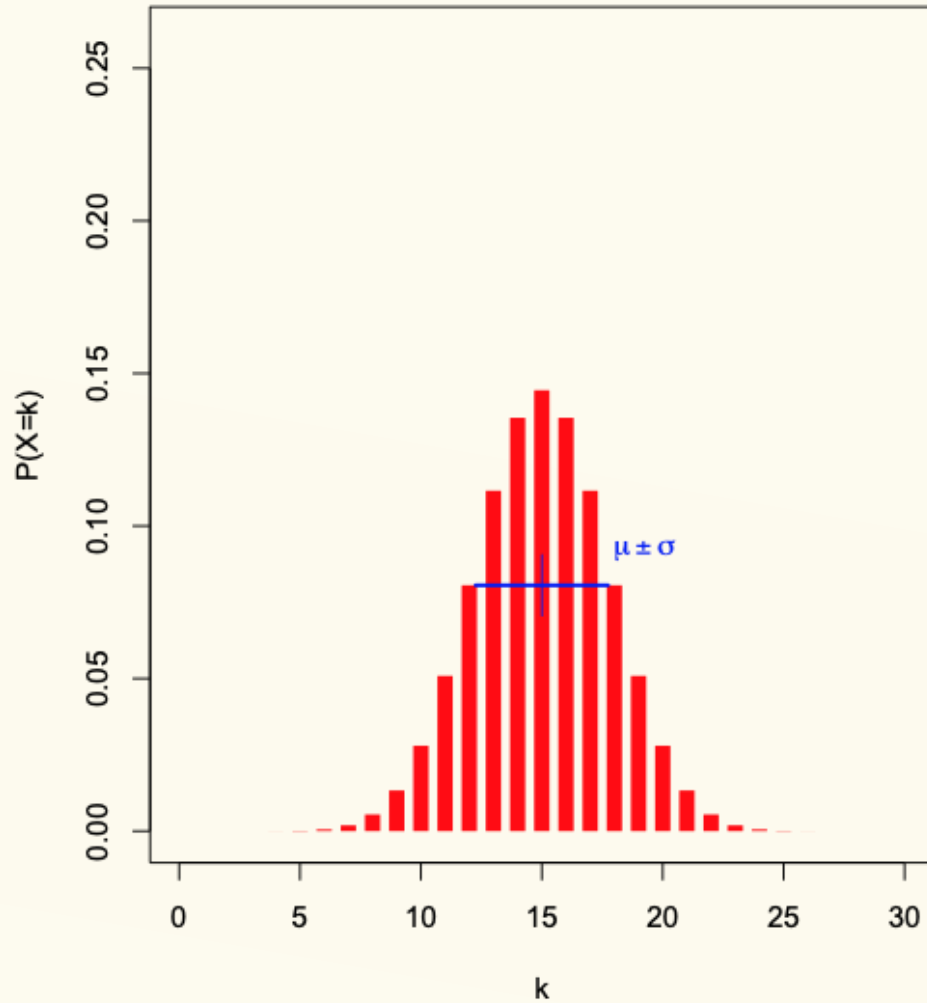


PMF for  $X \sim \text{Bin}(10,0.25)$

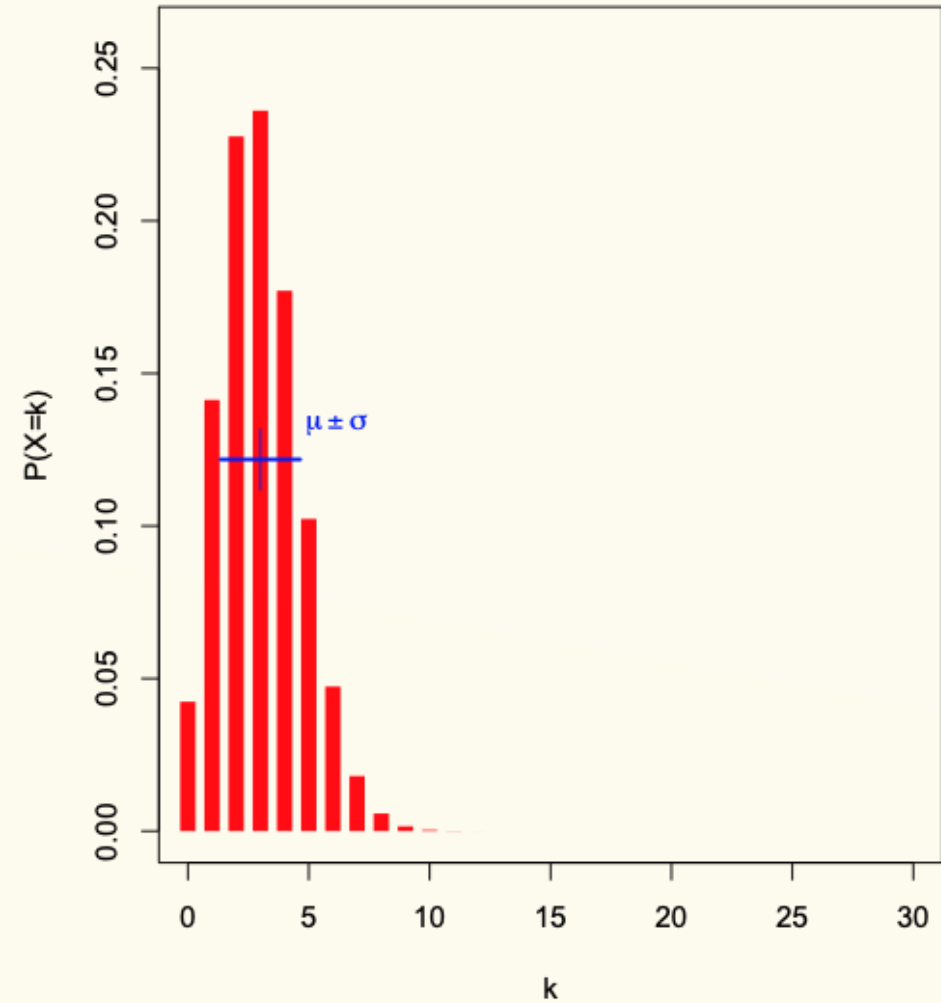


# Binomial PMFs

PMF for  $X \sim \text{Bin}(30, 0.5)$



PMF for  $X \sim \text{Bin}(30, 0.1)$






# Example

Sending a binary message of length 1024 bits over a network with probability 0.999 of correctly sending each bit in the message without corruption (independent of other bits). Let  $X$  be the number of corrupted bits. What is  $E[X]$ ?

Poll:

- a. 1022.99
- b. 1.024
- c. 1.02298
- d. 1
- e. Not enough information to compute

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- **Geometric Random Variables** 
  - Bonus material
- Poisson

# Geometric Random Variables

A discrete random variable  $X$  that models the number of independent trials  $Y_i \sim \text{Ber}(p)$  before seeing the first success.  $X$  is called a **Geometric random variable** with parameter  $p$ .

**Notation:**  $X \sim \text{Geo}(p)$

**PMF:**

**Expectation:**

**Variance:**

**Examples:**

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

# Geometric Random Variables

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**Notation:**  $X \sim \text{Geo}(p)$

**PMF:**  $\Pr(X = k) = (1 - p)^{k-1}p$

**Expectation:**  $E[X] = \frac{1}{p}$

**Variance:**  $\text{Var}(X) = \frac{1-p}{p^2}$

## Examples:

- # of coin flips until first head
- # of random guesses on MC questions until you get one right
- # of random guesses at a password until you hit it

## Example: Music Lessons

Your music teacher requires you to play a 1000 note song without mistake. You have been practicing, so you have a probability of 0.999 of getting each note correct (independent of the others). If you mess up a single note in the song, you must start over and play from the beginning. Let  $X$  be the number of times you have to play the song from the start. What is  $E[X]$ ?

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## Bonus: Negative Binomial Random Variables

A discrete random variable  $X$  that models the number of independent trials  $Y_i \sim \text{Ber}(p)$  before seeing the  $r^{\text{th}}$  success. Equivalently,  $X = \sum_{i=1}^r Z_i$  where  $Z_i \sim \text{Geo}(p)$ .  $X$  is called a **Negative Binomial random variable** with parameters  $r, p$ .

**Notation:**  $X \sim \text{NegBin}(r, p)$

**PMF:**

**Expectation:**

**Variance:**

## Bonus: Negative Binomial Random Variables

A discrete random variable  $X$  that models the number of independent trials  $Y_i \sim \text{Ber}(p)$  before seeing the  $r^{\text{th}}$  success. Equivalently,  $X = \sum_{i=1}^r Z_i$  where  $Z_i \sim \text{Geo}(p)$ .  $X$  is called a **Negative Binomial random variable** with parameters  $r, p$ .

**Notation:**  $X \sim \text{NegBin}(r, p)$

**PMF:**  $\Pr(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

**Expectation:**  $E[X] = \frac{r}{p}$

**Variance:**  $\text{Var}(X) = \frac{r(1-p)}{p^2}$



## Bonus: Hypergeometric Random Variables

A discrete random variable  $X$  that measures the number of white balls you draw when you draw  $n$  balls uniformly at random from a total of  $N$  of which  $K$  are white and the rest are black.  $X$  is called a **Hypergeometric RV** with parameters  $N, K, n$ .

**Notation:**  $X \sim \text{HypGeo}(N, K, n)$

**PMF:**

**Expectation:**

## Bonus: Hypergeometric Random Variables

A discrete random variable  $X$  that measures the number of white balls you draw when you draw  $n$  balls uniformly at random from a total of  $N$  of which  $K$  are white and the rest are black.  $X$  is called a **Hypergeometric RV** with parameters  $N, K, n$ .

**Notation:**  $X \sim \text{HypGeo}(N, K, n)$

**PMF:**  $\Pr(X = k) = \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}$

**Expectation:**  $E[X] = n \frac{K}{N}$

**Variance:**  $\text{Var}(X) = n \frac{K(N-K)(N-n)}{N^2(N-1)}$

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# Poisson Distribution

- Suppose “events” happen, independently, at an *average* rate of  $\lambda$  per unit time.
- Let  $X$  be the *actual* number of events happening in a given time unit. Then  $X$  is a *Poisson* r.v. with parameter  $\lambda$  (denoted  $X \sim \text{Poi}(\lambda)$ ) and has distribution (PMF):

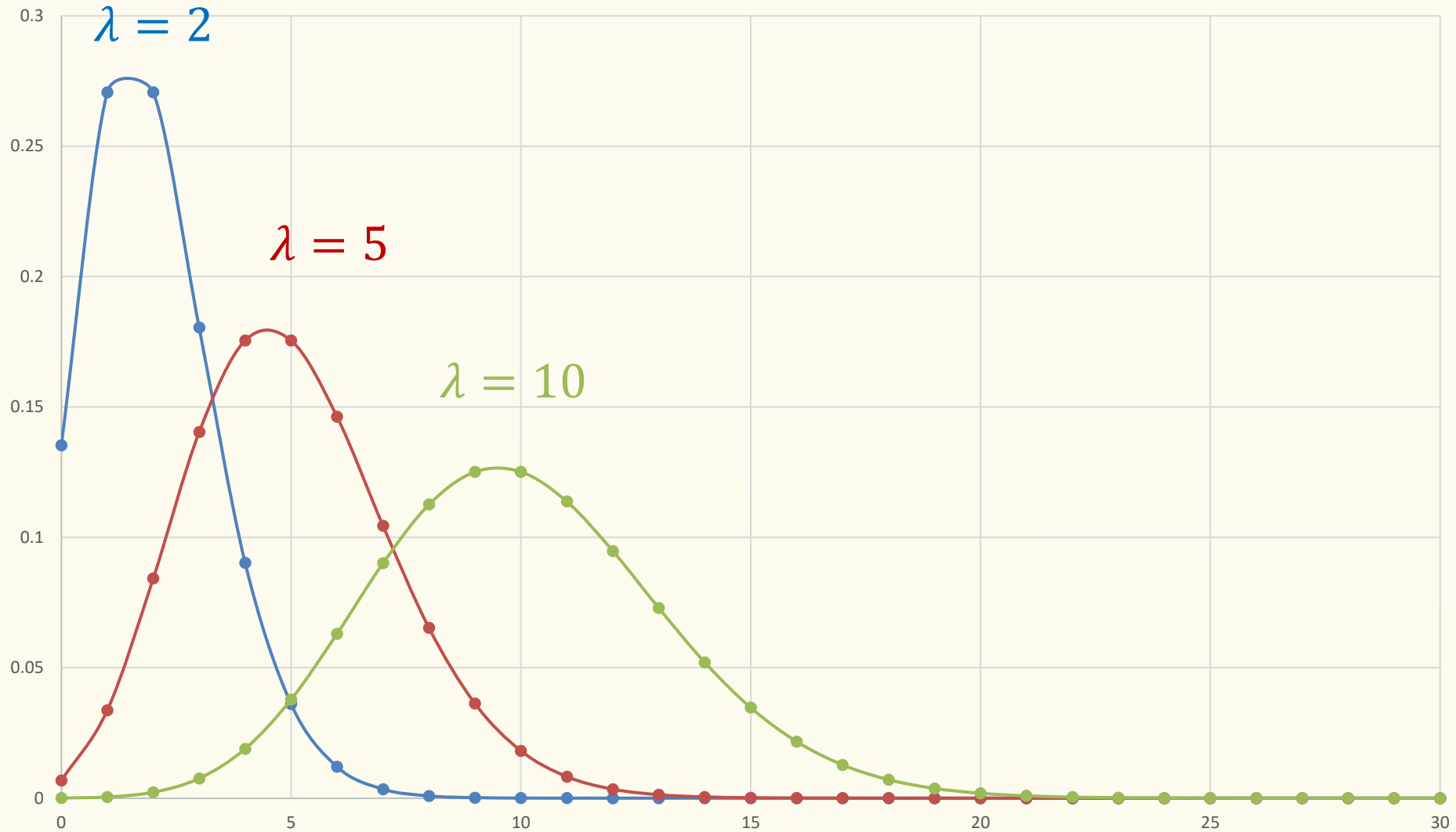
$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Several examples of “Poisson processes”:

- # of cars passing through a certain town in 1 hour
  - # of requests to web servers in a minute
  - # of photons hitting a light detector in a given interval
  - # of patients arriving to ER within an hour
- Assume fixed average rate

# Probability Mass Function

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



# Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) =$$

$$\text{Fact. } \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$$

# Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}_{= e^{\lambda}} = e^{-\lambda} e^{\lambda} = 1$$

$$\text{Fact. } \sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$$

## Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If  $X$  is a Poisson RV with parameter  $\lambda$ , then

$$\mathbb{E}(X) = \lambda$$

**Proof.**  $\mathbb{E}(X) = \sum_{i=0}^{\infty} i \cdot \mathbb{P}(X = i)$



# Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If  $X$  is a Poisson RV with parameter  $\lambda$ , then

$$\mathbb{E}(X) = \lambda$$

**Proof.**

$$\begin{aligned}\mathbb{E}(X) &= \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} \\ &= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \\ &= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda\end{aligned}$$

= 1 (see prior slides!)

# Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If  $X$  is a Poisson RV with parameter  $\lambda$ , then  $\text{Var}(X) = \lambda$

**Proof.** 
$$\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \lambda^2 + \lambda$$



$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

# Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Theorem.** If  $X$  is a Poisson RV with parameter  $\lambda$ , then  $\text{Var}(X) = \lambda$

**Proof.**

$$\begin{aligned}\mathbb{E}(X^2) &= \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i \\ &= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1) \\ &= \lambda \left[ \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}(X) = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda\end{aligned}$$

Similar to the previous proof  
Verify offline.



$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

# Poisson Random Variables

**Definition.** A **Poisson random variable**  $X$  with parameter  $\lambda \geq 0$  is such that for all  $i = 0, 1, 2, 3 \dots$ ,

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

**Notation:**  $X \sim \text{Poi}(\lambda)$

**PMF:**  $\Pr(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

**Expectation:**  $E[X] = \lambda$

**Variance:**  $\text{Var}(X) = \lambda$

# Sum of Independent Poisson RVs

**Theorem.** Let  $X \sim Poi(\lambda_1)$  and  $Y \sim Poi(\lambda_2)$  such that  $\lambda = \lambda_1 + \lambda_2$ .

Let  $Z = (X + Y)$ . For all  $k = 0, 1, 2, 3, \dots$ ,

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

More generally, let  $X_1 \sim Poi(\lambda_1), \dots, X_n \sim Poi(\lambda_n)$  such that  $\lambda = \sum_i \lambda_i$ .

Let  $Z = \sum_i X_i$

$$\mathbb{P}(Z = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

## Poisson Example

There are two ERs in a small town that act independently. The first has an average of 4 patients admitted per hour, and the second has an average of 3. What is the likelihood that in the next hour, 10 patients are admitted across both ERs?