CSE 312

Foundations of Computing II

Lecture 8: Introduction to Random Variables

PAUL G. ALLEN SCHOOL OF COMPUTER SCIENCE & ENGINEERING

Aleks Jovcic

Slide Credit: Based on Stefano Tessaro's slides for 312 19au

incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & Anna ☺

Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

- *What is the total of two dice rolls?*
- *What is the number of coin tosses needed to see the first head?*
- *What is the number of heads among 2 coin tosses?*

Random Variables

Definition. A **random variable (RV)** for a probability space $(\Omega, \mathbb{P})$ is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is called its range/support Ω_X

Example. Number of heads in 2 independent coin flips $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$

RV Example

20 balls labeled 1, 2, … , 20 in a bin

- Draw a subset of 3 uniformly at random
- Let $X = \text{maximum of the 3 numbers on the balls}$
  - Example: $X(2,7,5) = 7$
  - Example: $X(15,3,8) = 15$
- What is $|\Omega_X|$?

A. 20^3
B. 20
C. 18
D. $\binom{20}{3}$

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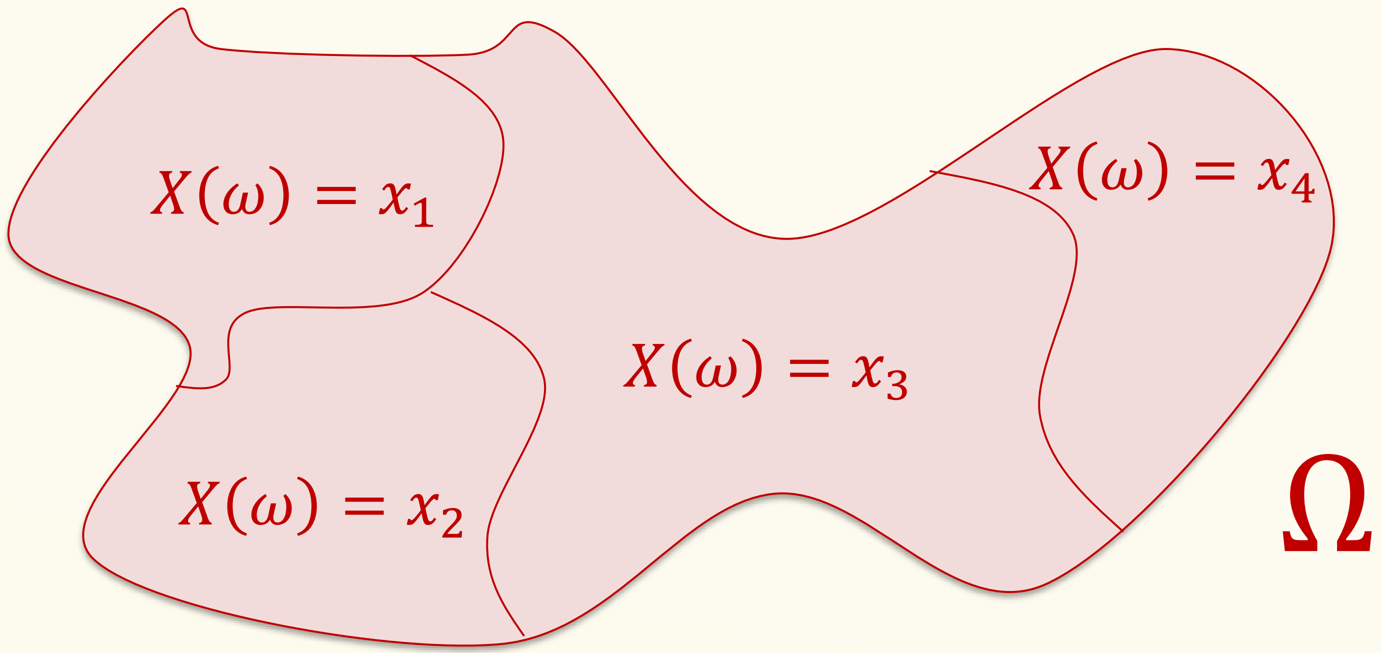
Probability Mass Function (PMF)

Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$



Probability Mass Function (PMF)

Definition. For a RV $X: \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} \stackrel{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $\mathbb{P}(X = x) = \mathbb{P}(\{X = x\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$ where $\mathbb{P}(X = x)$ is the probability mass function (PMF) of X

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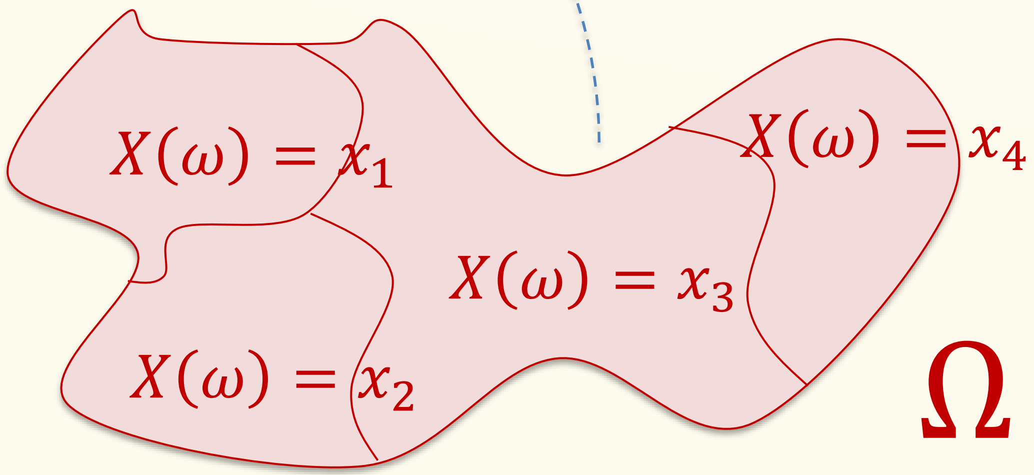

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Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$

You also see this notation (e.g. in book):

$$\mathbb{P}(\boldsymbol{X} = \boldsymbol{x}) = \boldsymbol{p}_{\boldsymbol{X}}(\boldsymbol{x})$$

Probability Mass Function

Flipping two independent coins

$$\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$$

$X = \text{number of heads in the two flips}$

$X(HH) = 2$ $X(HT) = 1$ $X(TH) = 1$ $X(TT) = 0$

$$\Omega_{\mathrm{X}} = \{0, 1, 2\}$$

What is $Pr(X = k)$?

RV Example

20 balls labeled 1, 2, …, 20 in a bin

- Draw a subset of 3 uniformly at random
- Let $X =$ maximum of the 3 numbers on the balls

What is $Pr(X = 20)$?

A. $\binom{20}{2} / \binom{20}{3}$

B. $\binom{19}{2} / \binom{20}{3}$

C. $19^2 / \binom{20}{3}$

D. $19 \cdot 18 / \binom{20}{3}$

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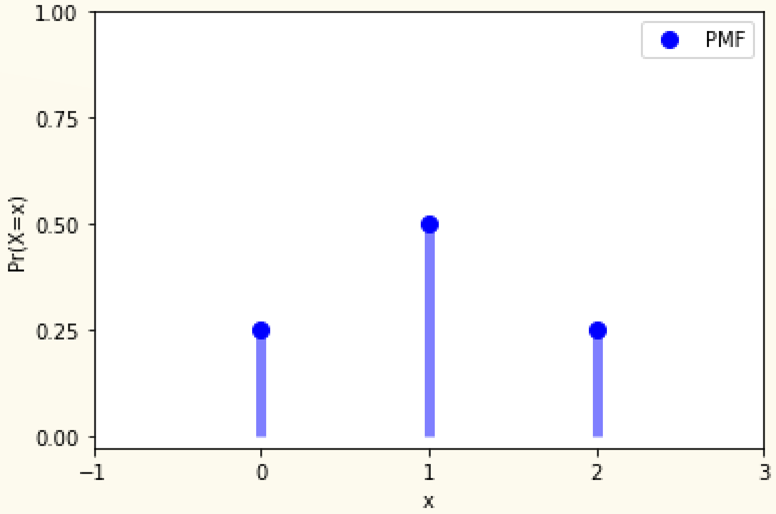
Cumulative Distribution Function (CDF)

Definition. For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of where X specifies for any real number x, the probability that $X \leq x$.

$$F_X(x) = \Pr(X \leq x)$$

Go back to 2 coin clips, where X is the number of heads

$$\Pr(X = x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases}$$



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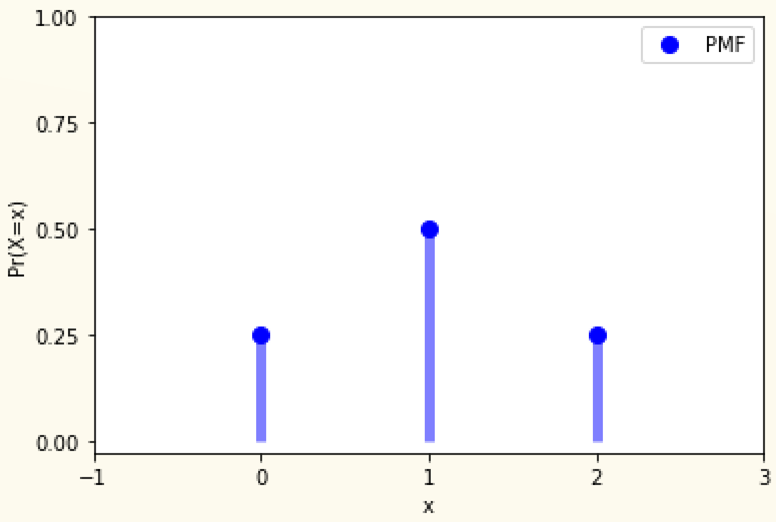
Go back to 2 coin clips, where X is the number of heads

$$\Pr(X = x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < 1 \\ \frac{3}{4}, & 1 \leq x < 2 \\ 1, & 2 \leq x \end{cases}$$



Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

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Expectation (Idea)

What is the *expected* number of heads in 2 independent flips of a fair coin?

Cumulative Disribution Function (CDF)

Definition. Given a discrete RV $X : \Omega \to \mathbb{R}$, the expectation or expected value of X is

$$\mathrm{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$$

or equivalently

$$\mathrm{E}[X] = \sum_{x \in X(\Omega)} x \cdot \Pr(X = x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

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1/6	3, 1, 2	0
1/6	3, 2, 1	1

Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability p of being heads. Keep flipping independent flips until heads. Let X be the number of flips until heads.

What is: $\Pr(X = 1) =$

What is: $\Pr(X = 2) =$

What is: $\Pr(X = k) =$

Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability p of being heads. Keep flipping independent flips until heads. Let X be the number of flips until heads. What is $\mathrm{E}[X]$?

Students on a bus

A group of 120 students are driven on 3 buses to a football game. There are 36 students in the first bus, 40 in the second and 44 in the third. Let Y be the number of students on a uniformly random bus. What is the pmf of Y and E(Y)? When the buses arrive, one of the 120 students is randomly chosen. Let X denote the number of students on the bus of the randomly chosen student. What is the pmf of X and what is E(X)?

Coin flipping again

Suppose we flip a coin with probability p of coming up Heads n times. Let X be the number of Heads in the n coin flips. What is the pmf of X?