CSE 312

Foundations of Computing II

Lecture 6: More Conditional Probability

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

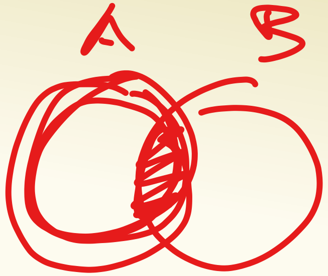
Agenda

- Review: Conditional Probability, Bayes
- Law of Total Probability (w/ Bayes)
- Chain Rule
- Independence
- Conditional Independence
- Assumptions and Correlation
- Bonus: Monty Hall Problem

Last Class:

- Conditional Probability
- Bayes Theorem

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})}$$



$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

$$\mathbb{P}(A|B)$$

$$\mathbb{P}(A|B) \overset{?}{\neq} \mathbb{P}(B|A)$$

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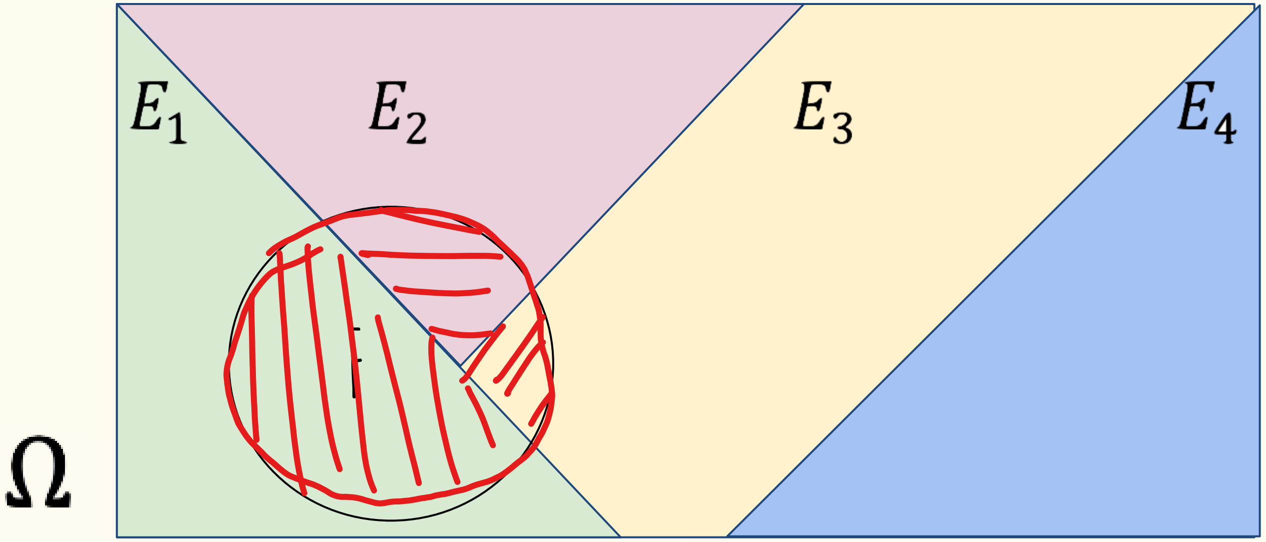
Law of Total Probability (Idea)

If we know $E_1, E_2, \ldots, E_n$ partition Ω, what can we say about $P(F)$



Law of Total Probability (LTP)

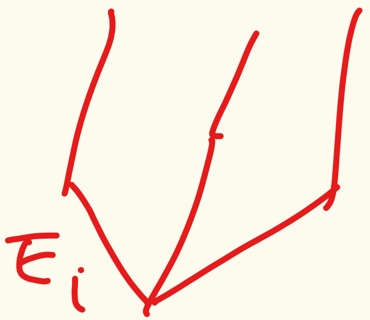
Definition. If events $E_1, E_2, \dots, E_n$ partition the sample space Ω, then for any event F

$$P(F) = P(F \cap E_1) + \dots + P(F \cap E_n) = \sum_{i=1}^{n} P(F \cap E_i)$$

Using the definition of conditional probability $P(F \cap E) = P(F|E)P(E)$

We can get the alternate form of this that show

$$P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$



Another Contrived Example

Alice has two pockets: 1/2

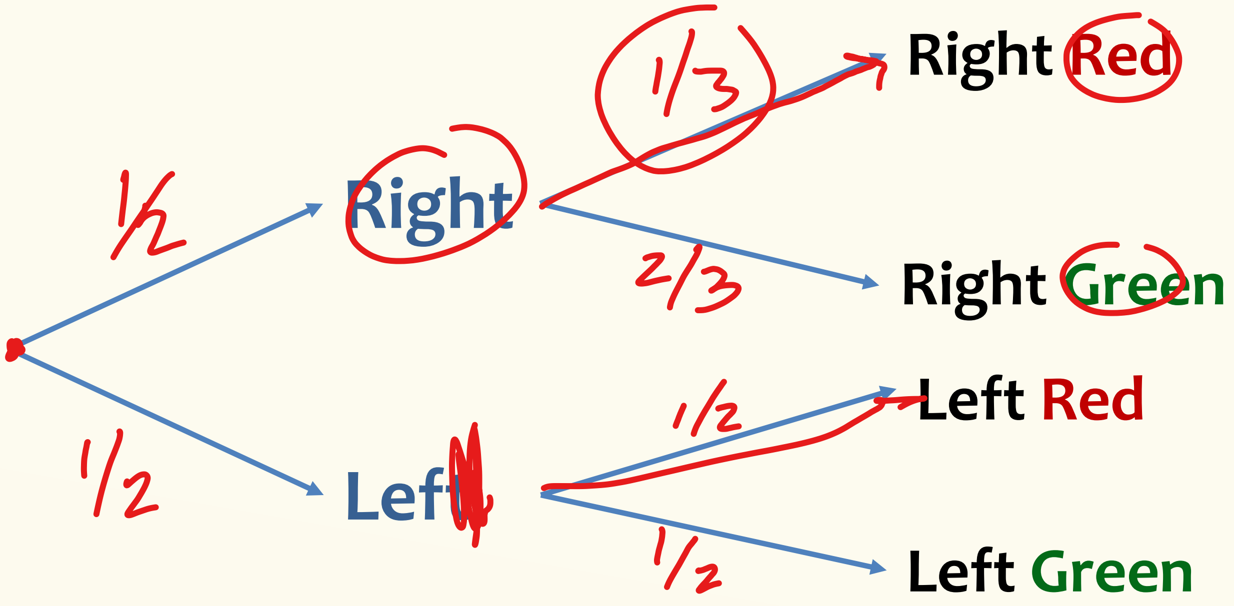
- **Left pocket:** Two red balls, two green balls
- **Right pocket:** One red ball, two green balls. 1/3

Alice picks a random ball from a random pocket.
[Both pockets equally likely, each ball equally likely.]

R = red ball

What is $\mathbb{P}(\mathbf{R})$?

Sequential Process – Non-Uniform Case



- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

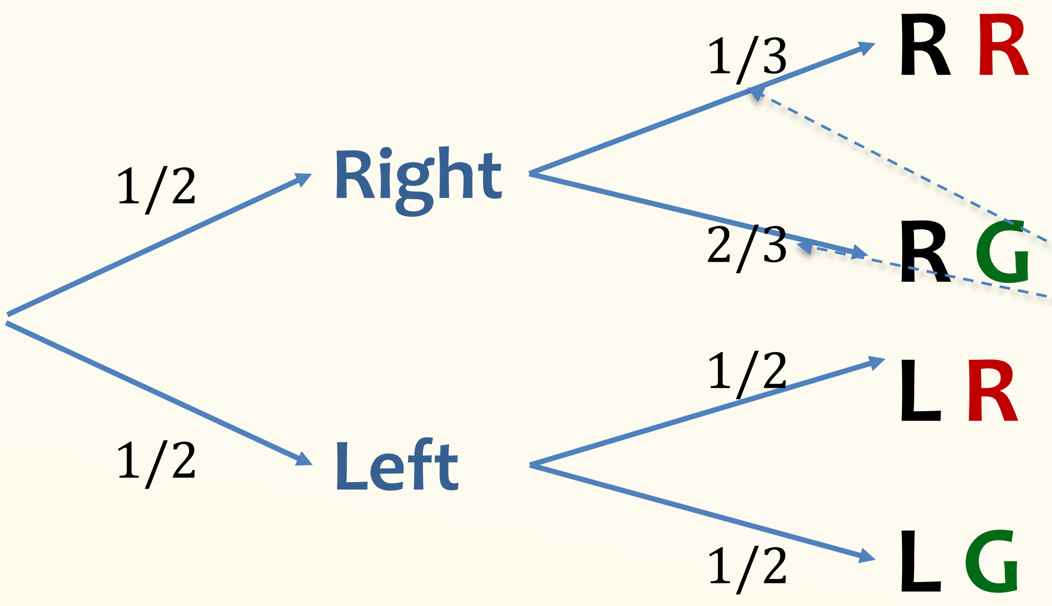
- Alice picks a random ball from a random pocket

$$\mathbb{P}(R) = \mathbb{P}(R \cap \text{Left}) + \mathbb{P}(R \cap \text{Right})$$

$$= \mathbb{P}(R \mid \text{Left})\,\mathbb{P}(\text{Left}) + \mathbb{P}(R \mid \text{Right})\,\mathbb{P}(\text{Right})$$

$$= \tfrac{1}{2} \cdot \tfrac{1}{2} + \tfrac{1}{3} \cdot \tfrac{1}{2} =$$

Sequential Process – Non-Uniform Case



- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

1/3 = $\mathcal{P}(R \mid \mathrm{R})$ and 2/3 = $\mathcal{P}(G \mid \mathrm{R})$

$$\mathbb{P}(\mathbf{R}) = \mathbb{P}(\mathbf{R} \cap \mathbf{Left}) + \mathbb{P}(\mathbf{R} \cap \mathbf{Right})$$

(Law of total probability)

$$= \mathbb{P}(\mathbf{Left}) \times \mathbb{P}(\mathbf{R}|\mathbf{Left}) + \mathbb{P}(\mathbf{Right}) \times \mathbb{P}(\mathbf{R}|\mathbf{Right})$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{4} + \frac{1}{6} = \boxed{\frac{5}{12}}$$

Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let $E_1, E_2, \dots, E_n$ be a partition of the sample space, and F and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

Simple Partition: In particular, if E is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

Example – Zika Testing

Usually no or mild symptoms (rash); sometimes severe symptoms (paralysis).

During pregnancy: may cause birth defects.

Suppose you took a Zika test, and it returns "positive", what is the likelihood that you actually have the disease?

- Tests for diseases are rarely 100% accurate.

This example and following slides are from Lisa Yan (Stanford).

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test yields a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event Z) if you test positive (event T).

$P(Z|T) = ?$

A) Less than 0.25
B) Between 0.25 and 0.5
C) Between 0.5 and 0.75
D) Between 0.75 and 1

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $\mathbb{P}(T \mid Z) = 0.98$
- However, the test yields a "false positive" 1% of the time $\mathbb{P}(T \mid Z^c) = 0.01$
- 0.5% of the US population has Zika. $\mathbb{P}(Z) = 0.005$

What is the probability you have Zika (event Z) if you test positive (event T).

$$\mathbb{P}(Z \mid T) = ? = \frac{\mathbb{P}(T \mid Z)\mathbb{P}(Z)}{\mathbb{P}(T)} = \frac{\mathbb{P}(T \mid Z)\mathbb{P}(Z)}{\mathbb{P}(T \mid Z)\mathbb{P}(Z) + \mathbb{P}(T \mid Z^c)\mathbb{P}(Z^c)}$$

$$= \frac{0.98 \cdot 0.005}{0.98 \cdot 0.005 + 0.01 \cdot 0.995} = \boxed{0.33}$$

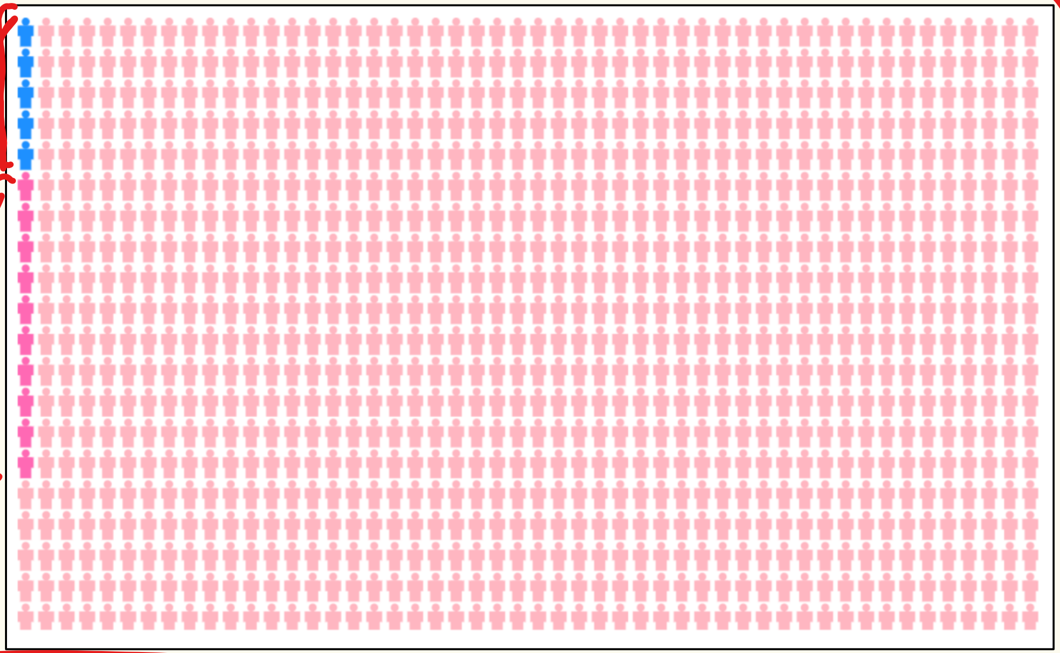
Example – Zika Testing

Have zika blue, don't pink

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") 100%
- However, the test may yield a "false positive" 1% of the time 10/995 = approximately 1%
- 0.5% of the US population has Zika. 5 people have it.

What is the probability you have Zika (event *Z*) if you test positive (event *T*).



Suppose we had 1000 people:

- 5 **have Zika** and **test positive**
- 985 **do not have Zika** and **test negative**
- 10 **do not have Zika** and **test positive**

$$\frac{5}{5+10} = \frac{1}{3} \approx 0.33$$

Demo

Philosophy – Updating Beliefs

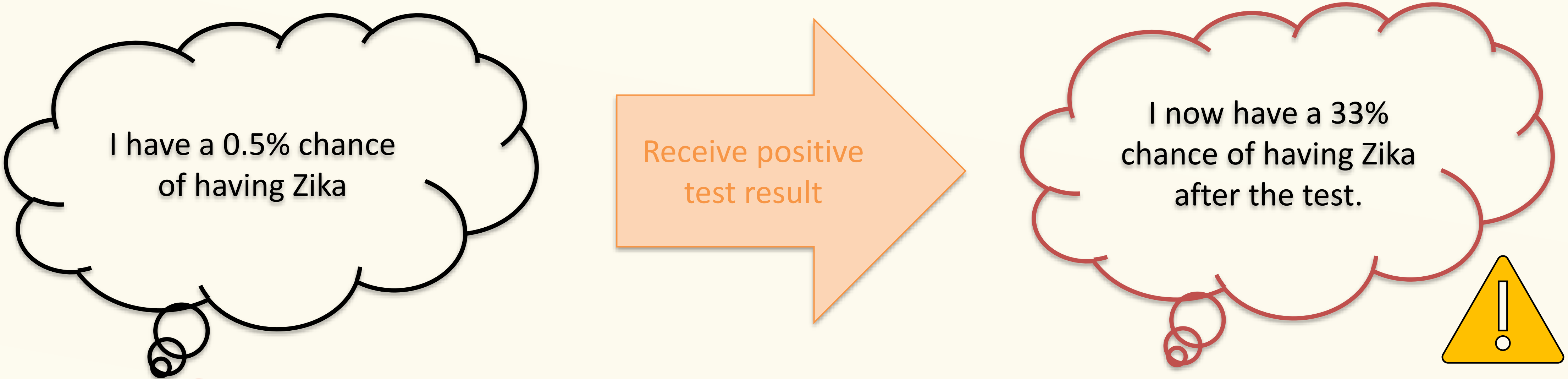
$P(Z \mid Z \cap T)$
$P(Z \mid Z, T)$

While it's not 98% that you have the disease, your beliefs changed **drastically**

$P(Z \mid T \cap T)$

Z = you have Zika

T = you test positive for Zika



Prior: P(Z)

Posterior: P(Z|T)

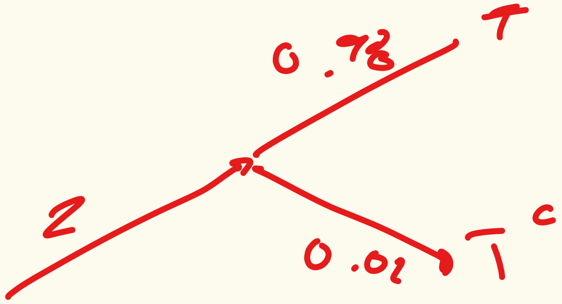
Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $P(T|Z) = 0.98$
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you test negative (event $\bar{T}$) if you have Zika (event Z)?

$P(T^c|Z) = 0.02$



$P(T^c|Z) = 1 - P(T|Z)$

$1 - 0.98$

Conditional Probability Define a Probability Space

The probability conditioned on A follows the same properties as (unconditional) probability.

$\mathbb{P}(B^c) = 1 - \mathbb{P}(B)$

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Conditional Probability Define a Probability Space

The probability conditioned on A follows the same properties as (unconditional) probability.

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Formally. $(\Omega, \mathbb{P})$ is a probability space + $\mathbb{P}(\mathcal{A}) > 0$

⇒ $(\mathcal{A}, \mathbb{P}(\cdot|\mathcal{A}))$ is a probability space

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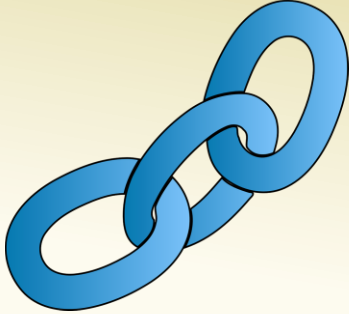
Chain Rule

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} \quad \Rightarrow \quad \mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B}|\mathcal{A})$$

$$\mathbb{P}(B \cap A) = \mathbb{P}(B)\mathbb{P}(A|B)$$

Chain Rule



$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} \Rightarrow \mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A})\mathbb{P}(\mathcal{B}|\mathcal{A})$$

Theorem. (Chain Rule) For events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$,

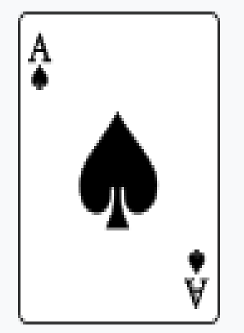
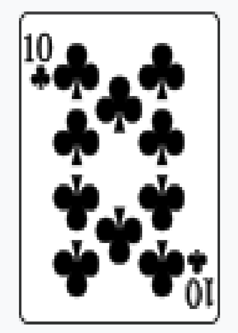
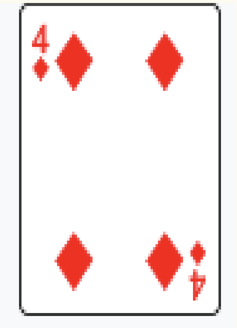
$$\mathbb{P}(\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_n) = \mathbb{P}(\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_2|\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_3|\mathcal{A}_1 \cap \mathcal{A}_2)$$

$$\cdots \mathbb{P}(\mathcal{A}_n|\mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_{n-1})$$

An easy way to remember: We have n tasks and we can do them sequentially, conditioning on the outcome of previous tasks

Chain Rule Example

Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards **in order**. (uniform probability space).

What is P (  ) = P(**A** ∩ **B** ∩ **C**)?

A: Ace of Spades First

B: 10 of Clubs Second

C: 4 of Diamonds Third

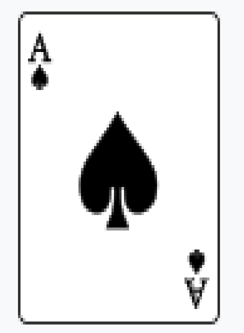
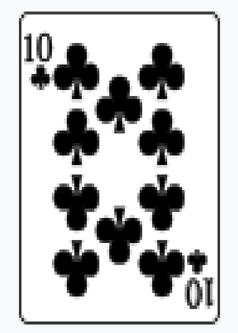
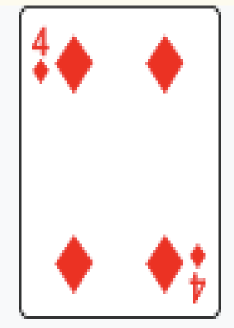
$P(A \cap B \cap C) = P(A) \cdot P(B \mid A) \cdot P(C \mid B \cap A)$

$= \frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$

$\frac{1}{P(52, 3)}$

Chain Rule Example

Have a Standard 52-Card Deck. Shuffle It, and draw the top 3 cards in order. (uniform probability space).

What is P (  ) = P(**A** ∩ **B** ∩ **C**)?

A: Ace of Spades First

B: 10 of Clubs Second

C: 4 of Diamonds Third

$$\mathbb{P}(A) \cdot \mathbb{P}(B|A) \cdot \mathbb{P}(C|A \cap B)$$

$$\frac{1}{52} \cdot \frac{1}{51} \cdot \frac{1}{50}$$

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Independence

$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B|A)$

$\mathbb{P}(A) \cdot \mathbb{P}(B)$

Definition. If two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** then

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

Alternatively,

- If $\mathbb{P}(\mathcal{A}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(\mathcal{B})$
- If $\mathbb{P}(\mathcal{B}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$

"The probability that $\mathcal{B}$ occurs after observing $\mathcal{A}$" -- Posterior
= "The probability that $\mathcal{B}$ occurs" -- Prior

Example -- Independence

Toss a coin 3 times. Each of 8 outcomes equally likely.

- A = {at most one T} = {~~HHH~~, HHT, HTH, THH}
- B = {at most 2 Heads}= $\{HHH\}^c$

Independent?

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) \overset{?}{=} \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B})$$

$\mathbb{P}(A) = \frac{1}{2}$

$\mathbb{P}(B) = \frac{7}{8}$

$\mathbb{P}(A \cap B) = \frac{3}{8}$

Poll:

A. Yes, independent

B. No

Often probability space $(\Omega, \mathbb{P})$ is **defined** using independence

$0 = 0 \cdot \frac{1}{2}$

Events generated independently ➔ *their probabilities satisfy independence*

~~⇐~~ *Not necessarily*

$\mathbb{P}(A) = \frac{1}{2}$

$\mathbb{P}(B) = 0$

$\mathbb{P}(A \cap B) = 0$

This can be counterintuitive!

Example – Network Communication

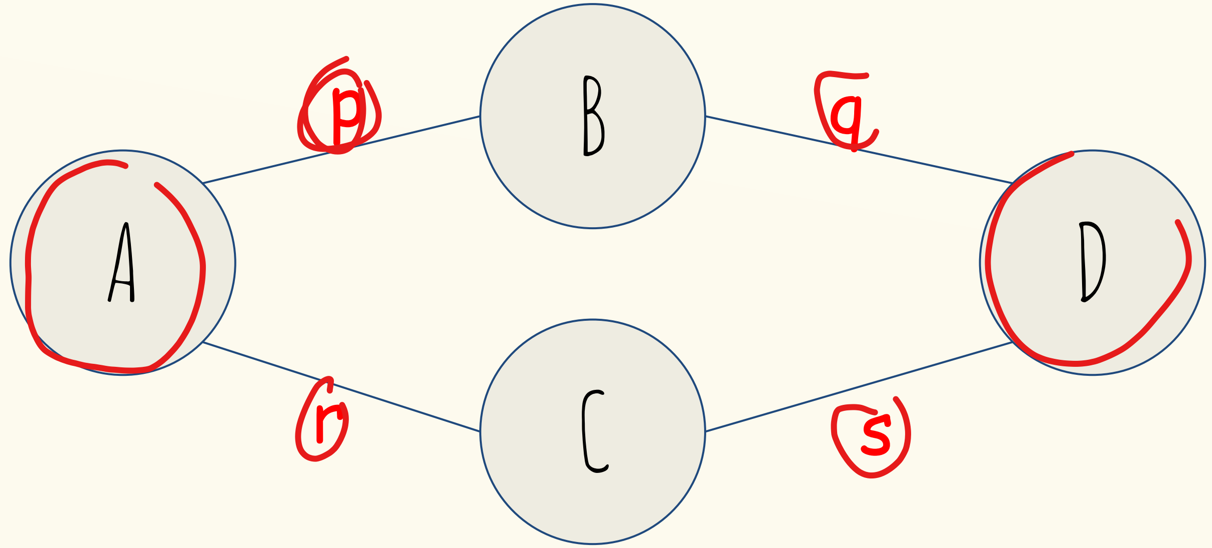
Each link works with the probability given, **independently**.
What's the probability A and D can communicate?

$\mathbb{P}(AD) = ?$ $\mathbb{P}((AB \cap BD) \cup (AC \cap CD))$

$= \mathbb{P}(AB \cap BD) + \mathbb{P}(AC \cap CD) - \mathbb{P}(AB \cap BD \cap AC \cap CD)$

$\mathbb{P}(AB)\mathbb{P}(BD)$

$pq + rs - pqrs$



Example – Network Communication

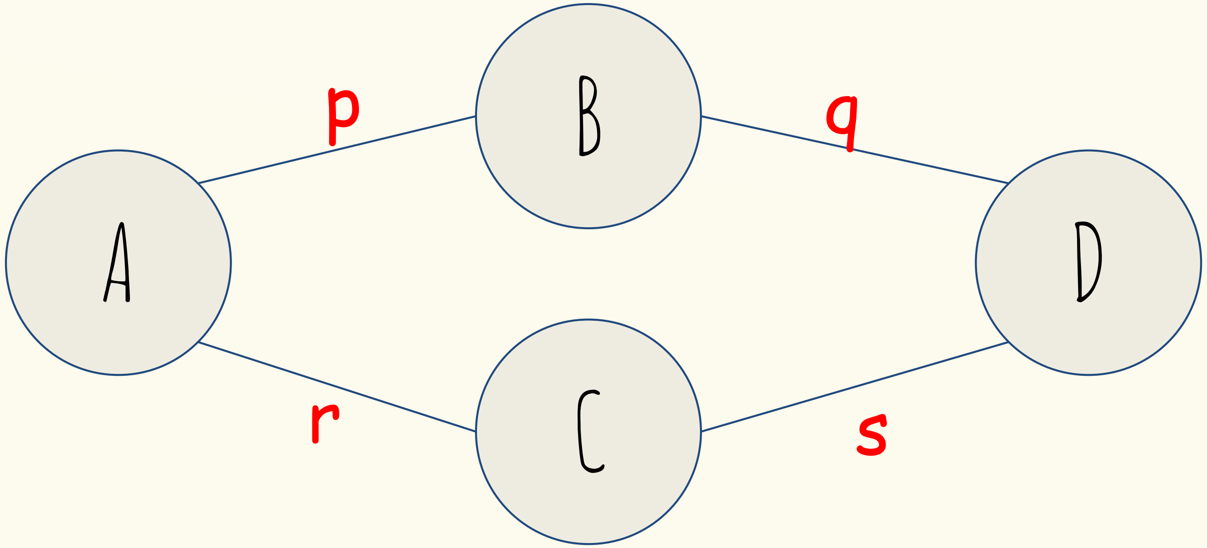
Each link works with the probability given, **independently**.
What's the probability A and D can communicate?

$$\mathbb{P}(AD) = \mathbb{P}(AB \cap BD \text{ } or \text{ } AC \cap CD)$$

$$= \mathbb{P}(AB \cap BD) + \mathbb{P}(AC \cap CD) - \mathbb{P}(AB \cap BD \cap AC \cap CD)$$

$$\mathbb{P}(AB \cap BD) = \mathbb{P}(AB) \cdot \mathbb{P}(BD) = pq$$

$$\mathbb{P}(AC \cap CD) = \mathbb{P}(AC) \cdot \mathbb{P}(CD) = rs$$



$$\mathbb{P}(AB \cap BD \cap AC \cap CD) = \mathbb{P}(AB) \cdot \mathbb{P}(BD) \cdot \mathbb{P}(AC) \cdot \mathbb{P}(CD) = pqrs$$

Example – Biased coin

We have a biased coin comes up Heads with probability 2/3; Each flip is independent of all other flips. Suppose it is tossed 3 times.

$\mathbb{P}(HHH) =$

$\mathbb{P}(TTT) =$

will go over next lecture

$\mathbb{P}(HTT) =$

Example – Biased coin

We have a biased coin comes up Heads with probability 2/3, independently of other flips. Suppose it is tossed 3 times.

$\mathbb{P}(2\ heads\ in\ 3\ tosses) =$

A) $(2/3)^2$ 1/3
B) 2/3
C) 3 $(2/3)^2$ 1/3
D) $(1/3)^2$

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- **Conditional Independence**
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ended here for today

Conditional Independence

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** conditioned on C if

$$\mathbb{P}(C) \neq 0 \text{ and } \mathbb{P}(\mathcal{A} \cap \mathcal{B} \mid C) = \mathbb{P}(\mathcal{A} \mid C) \cdot \mathbb{P}(\mathcal{B} \mid C).$$

Plain Independence. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

Equivalence:

- If $\mathbb{P}(\mathcal{A}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(B)$
- If $\mathbb{P}(\mathcal{B}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$

Conditional Independence

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** conditioned on C if

$$\mathbb{P}(C) \neq 0 \text{ and } \mathbb{P}(\mathcal{A} \cap \mathcal{B} \mid C) = \mathbb{P}(\mathcal{A} \mid C) \cdot \mathbb{P}(\mathcal{B} \mid C).$$

Equivalence:

- If $\mathbb{P}(\mathcal{A} \cap C) \neq 0$, equivalent to $\mathbb{P}(\mathcal{B}|\mathcal{A} \cap C) = \mathbb{P}(B \mid C)$
- If $\mathbb{P}(\mathcal{B} \cap C) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B} \cap C) = \mathbb{P}(\mathcal{A} \mid C)$

Plain Independence. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

Equivalence:

- If $\mathbb{P}(\mathcal{A}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{B}|\mathcal{A}) = \mathbb{P}(B)$
- If $\mathbb{P}(\mathcal{B}) \neq 0$, equivalent to $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A})$

Example – More coin tossing

Suppose there is a coin C1 with Pr(Head) = 0.3 and a coin C2 with Pr(Head) = 0.9. We pick one randomly with equal probability and flip that coin twice independently. What is the probability we get all heads?

$$\Pr(HH) = \Pr(HH \mid C1)\Pr(C1) + \Pr(HH \mid C2)\Pr(C2)$$

LTP

Example – More coin tossing

Suppose there is a coin C1 with Pr(Head) = 0.3 and a coin C2 with Pr(Head) = 0.9. We pick one randomly with equal probability and flip that coin 2 times independently. What is the probability we get all heads?

$$\Pr(HH) = \Pr(HH \mid C1)\Pr(C1) + \Pr(HH \mid C2)\Pr(C2)$$ LTP

$$= \Pr(H \mid C2)^2 \Pr(C1) + \Pr(H \mid C2)^2 \Pr(C2)$$ Conditional Independence

$$= 0.3^2 \cdot 0.5 + 0.9^2 \cdot 0.5 = 0.45$$

$$\Pr(H) = \Pr(H \mid C1)\Pr(C1) + \Pr(H \mid C2)\Pr(C2) = 0.6$$

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Correlation

- Pick a person at random
- A : event that the person has lung cancer
- B : event that the person is a heavy smoker

- Fact: $\mathbb{P}(A|B) = 1.17 \cdot \mathbb{P}(A)$

- Conclusions?

Correlation

- Pick a person at random
- A : event that the person has lung cancer
- B : event that the person is a heavy smoker

- Fact: $\mathbb{P}(A|B) = 1.17 \cdot \mathbb{P}(A)$

- Conclusions?
  - Lung cancer increases the the probability of smoking by 17%.
  - Lung cancer causes smoking.

Causality vs. Correlation

- Events A and B are **positively correlated** if

$$\mathbb{P}(A \cap B) > \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- E.g. smoking and lung cancer.

- But A and B being positively correlated does not mean that A causes B or B causes A.

Causality vs. Correlation

- Events A and B are **positively correlated** if

$$\mathbb{P}(A \cap B) > \mathbb{P}(A) \cdot \mathbb{P}(B)$$

- But A and B being positively correlated does not mean that A causes B or B causes A.

Other examples:

- Tesla owners are more likely to be rich. That does not mean poor people should buy a Tesla to get rich.
- People who go to the opera are more likely to have a good career. That does not mean that going to the opera will improve your career.
- Rabbits eat more carrots and do not wear glasses. Are carrots good for eyesight?

Independence as an assumption

- People often assume it **without justification.**
- Example: A sky diver has two chutes

A : event that the main chute doesn't open $\mathbb{P}(A) = 0.02$

B : event that the backup doesn't open $\mathbb{P}(B) = 0.1$

- What is the chance that at least one opens assuming independence?

Independence as an assumption

- People often assume it **without justification.**
- Example: A sky diver has two chutes

A : event that the main chute doesn't open $\mathbb{P}(A) = 0.02$

B : event that the backup doesn't open $\mathbb{P}(B) = 0.1$

- What is the chance that at least one opens assuming independence?

- Assuming independence doesn't justify the assumption! Both chutes could fail because of the same rare event e.g., freezing rain.

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Monty Hall Problem

Suppose you're on a game show, and you're given the choice of three doors. Behind one of the doors is a car, behind the other, goats. You pick a door, say number 1, and the host, who knows what's behind the doors, opens another door, say number 3, which has a goat. He says to you, "Do you want to switch to door number 2?" Is it to your advantage to switch your choice of doors?

Assumptions

- The player is equally likely to pick each of the three doors.
- After the player picks a door, the host must open a different door with a goat behind it and offer the player the choice of staying with the original door or switching.
- If the host has a choice of which door to open, then he is equally likely to select each of them.

Should you switch or stay?