CSE 312

# Foundations of Computing II

## Lecture 5: Conditional Probability Introduction

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Aleks Jovcic

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

# Belonging and CS Tas Research Study

Leah Perlmutter (she/her)
leahperl@uw.edu
tinyurl.com/belonging-study

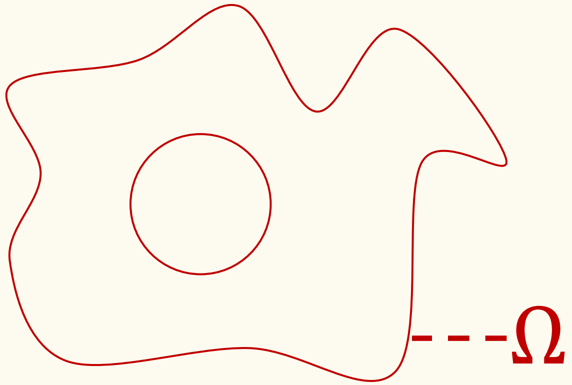
# Review Probability space

Either finite or infinite countable (e.g., integers)

**Definition.** A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

- $\Omega$ is a set called the **sample space.**
- $\mathbb{P}$ is the **probability measure,** a function $\mathbb{P}: \Omega \to [0,1]$ such that:
  - $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Set of possible **elementary outcomes**




Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

# Review Axioms of Probability

Let $\Omega$ denote the sample space and $E, F \subseteq \Omega$ be events. Note this is more general to **any** probability space (not just uniform)

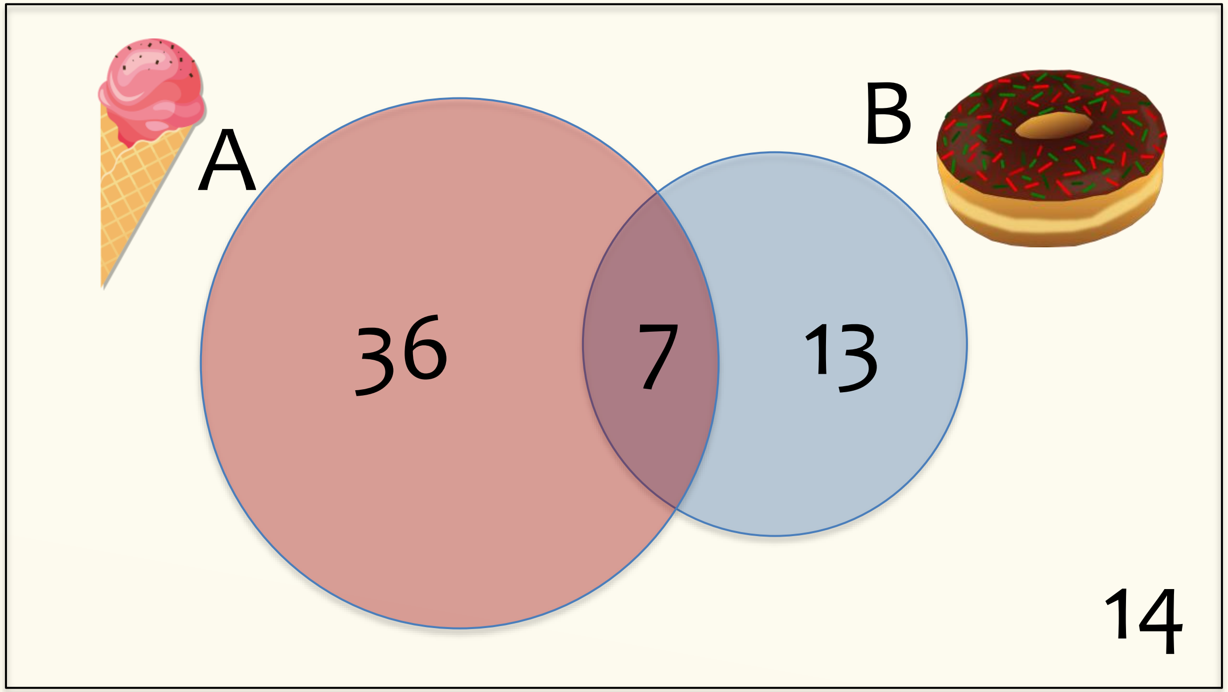
**Axiom 1 (Non-negativity):** $P(E) \geq 0$
**Axiom 2 (Normalization):** $P(\Omega) = 1$
**Axiom 3 (Countable Additivity)**: If $E$ and $F$ are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

**Corollary 1 (Complementation):** $P(E^c) = 1 - P(E)$
**Corollary 2 (Monotonicity):** If $E \subseteq F$, $P(E) \leq P(F)$
**Corollary 3 (Inclusion-Exclusion)**: $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

# Agenda

- Conditional Probability
- Time Permitting:
  - – Bayes Theorem
  - – Law of Total Probability
  - – Bayes Theorem + Law of Total Probability
  - – More Examples

# Conditional Probability (Idea)




What's the probability that someone likes ice cream **given** they like donuts?

# Conditional Probability

**Definition.** The **conditional probability** of event $A$ **given** an event $B$ happened (assuming $P(B) \neq 0$) is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

An equivalent and useful formula is

$$P(A \cap B) = P(A|B)P(B)$$

# Reversing Conditional Probability

**Question:** Does $P(A|B) = P(B|A)$?

No! The following is purely for intuition and makes no sense in terms of probability

- Let A be the event you are wet
- Let B be the event you are swimming

$$P(A|B) = 1$$
$$P(B|A) \neq 1$$

# Example with Conditional Probability

Toss a red die and a blue die (both 6 sided and all outcomes equally likely). What is $P(B)$? What is $P(B|A)$?

| | $\mathrm{P}(B)$ | $P(B|A)$ |
|---|---|---|
| a) | 1/6 | 1/6 |
| b) | 1/6 | 1/3 |
| c) | 1/6 | 3/36 |
| d) | 1/9 | 1/3 |

B = red die is 1

A = sum is 4

# Example with Conditional Probability

Toss a red die and a blue die (both 6 sided and all outcomes equally likely). What is $P(B)$? What is $P(B|A)$?

| | $\mathrm{P}(B)$ | $P(B|A)$ |
|---|---|---|
| a) | 1/6 | 1/6 |
| b) | 1/6 | 1/3 |
| c) | 1/6 | 3/36 |
| d) | 1/9 | 1/3 |

B = red die is 1

A = sum is 4

$$\mathbb{P}(B) = \frac{1}{6}$$

$$\mathbb{P}(B|A) = \frac{1}{3}$$

# Gambler's fallacy

Assume we toss **51** fair coins.
Assume we have seen **50** coins, and they are all "tails".
What are the odds the **51**$^{st}$ coin is "heads"?

$\mathcal{A}$ = first 50 coins are "tails"

$B$ = 51$^{st}$ coin is "heads"

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) =$$

# Gambler's fallacy

Assume we toss **51** fair coins.
Assume we have seen **50** coins, and they are all "tails".
What are the odds the **51st** coin is "heads"?

$\mathcal{A}$ = first 50 coins are "tails"

$B$ = 51st coin is "heads"

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} = \frac{1/2^{51}}{2/2^{51}} = \frac{1}{2}$$

51st coin is independent of outcomes of first 50 tosses!

**Gambler's fallacy** = Feels like it's time for " heads"!?

# Agenda

- Conditional Probability
- Time Permitting:
  - Bayes Theorem
  - Law of Total Probability
  - Bayes Theorem + Law of Total Probability
  - More Examples

# Bayes Theorem

A formula to let us "reverse" the conditional.

**Theorem. (Bayes Rule)** For events $A$ and $B$, where $P(A), P(B) > 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$P(A)$ is called the **prior** (our belief without knowing anything)

$P(A|B)$ is called the **posterior** (our belief after learning $B$)

# Bayes Theorem Proof

By definition of conditional probability

$$P(A \cap B) = P(A|B)P(B)$$

Swapping A, B gives

$$P(B \cap A) = P(B|A)P(A)$$

But $P(A \cap B) = P(B \cap A)$, so

$$P(A|B)P(B) = P(B|A)P(A)$$

Dividing both sides by $P(B)$ gives

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

# Our First Machine Learning Task: Spam Filtering

Subject: "FREE $$$ CLICK HERE"

What is the probability this email is spam, given the subject contains "FREE"?
Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word "FREE" in the subject.
- 70% of spam emails contain the word "FREE" in the subject.
- 80% of emails you receive are spam.

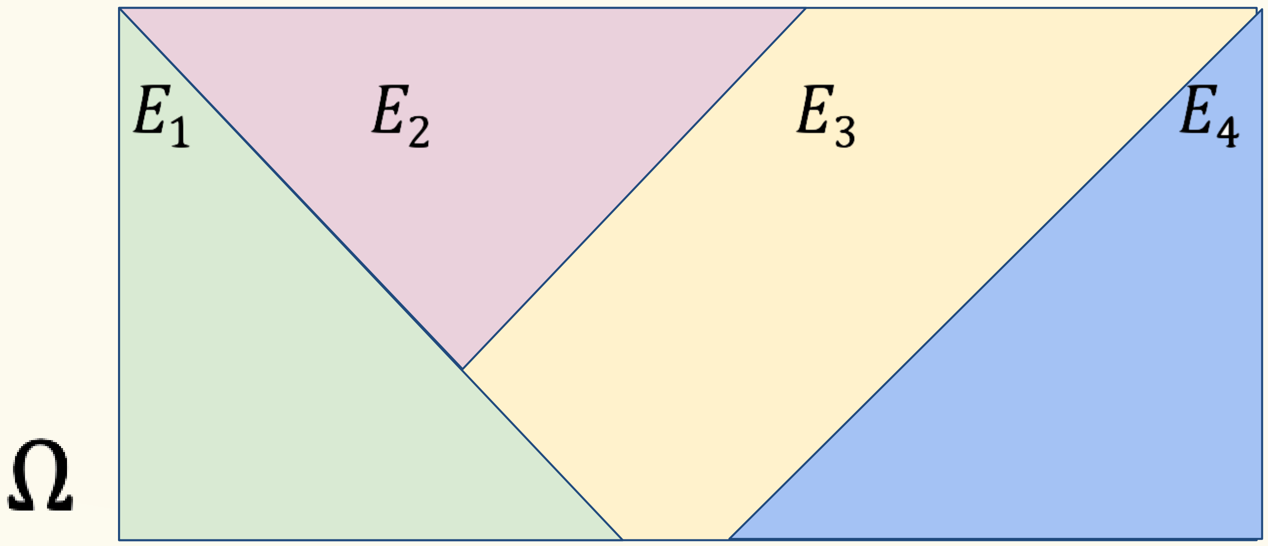
# Agenda

- Conditional Probability
- Time Permitting:
  - Bayes Theorem
  - Law of Total Probability
  - Bayes Theorem + Law of Total Probability
  - More Examples

# Partitions (Idea)

These events **partition** the sample space

1. They "cover" the whole space
2. They don't overlap



# Partition

**Definition.** Non-empty events $E_1, E_2, \ldots, E_n$ **partition** the sample space $\Omega$ if

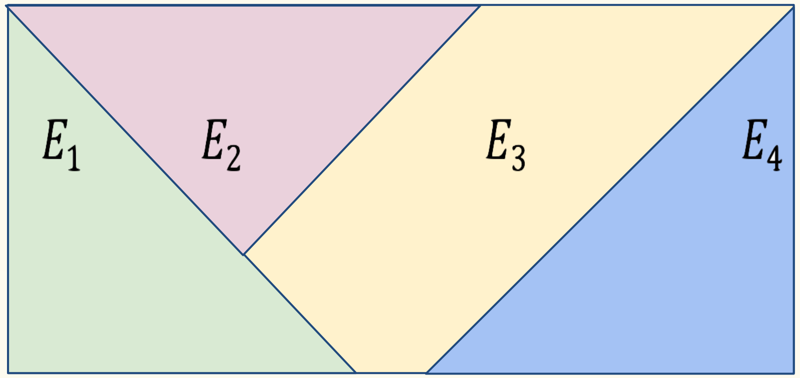
**(Exhaustive)**

$$E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$$

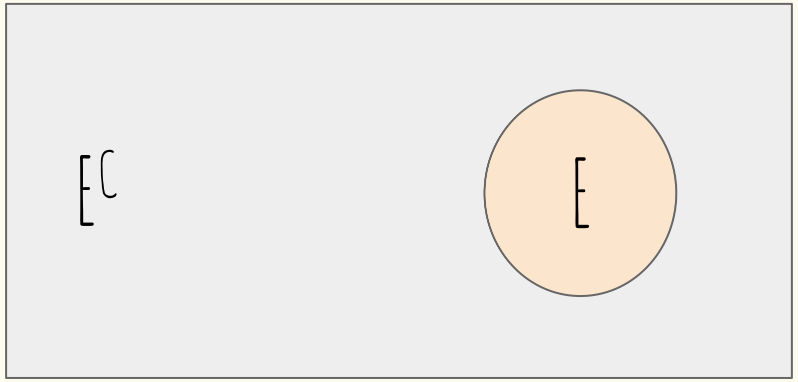
**(Pairwise Mutually Exclusive)**

$$\forall_i \forall_{i \neq j} E_i \cap E_j = \emptyset$$

$\Omega$

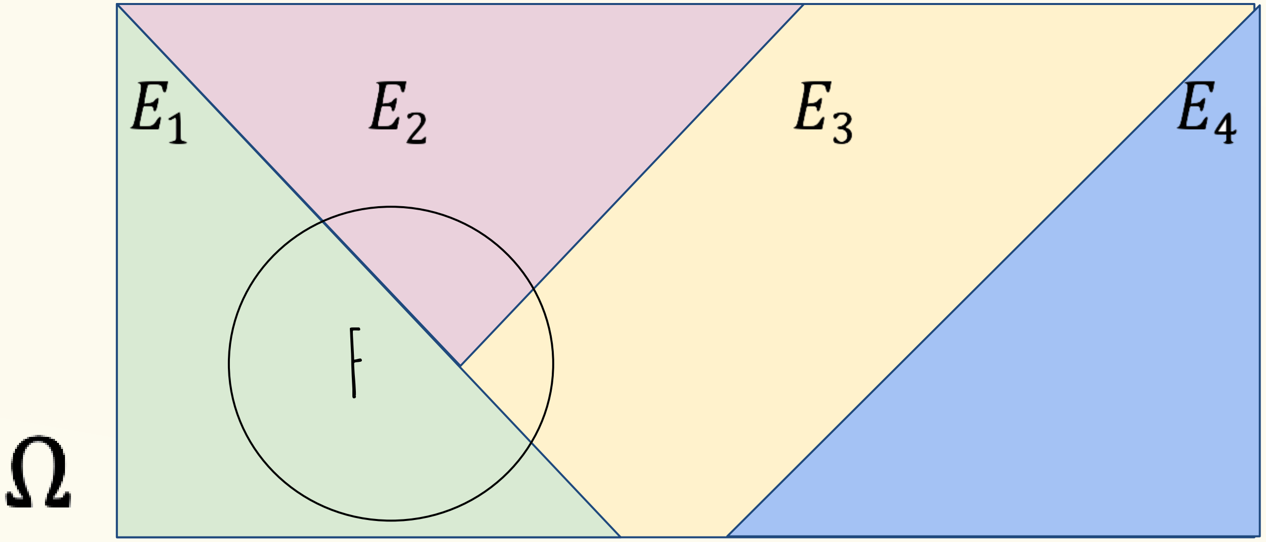



$\Omega$



# Law of Total Probability (Idea)

If we know $E_1, E_2, \dots, E_n$ partition $\Omega$, what can we say about $P(F)$



# Law of Total Probability (LTP)

**Definition.** If events $E_1, E_2, \dots, E_n$ partition the sample space $\Omega$, then for any event $\mathrm{F}$

$$P(F) = P(F \cap E_1) + \dots + P(F \cap E_n) = \sum_{i=1}^{n} P(F \cap E_i)$$

Using the definition of conditional probability $P(F \cap E) = P(F|E)P(E)$
We can get the alternate form of this that show

$$P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

# Another Contrived Example

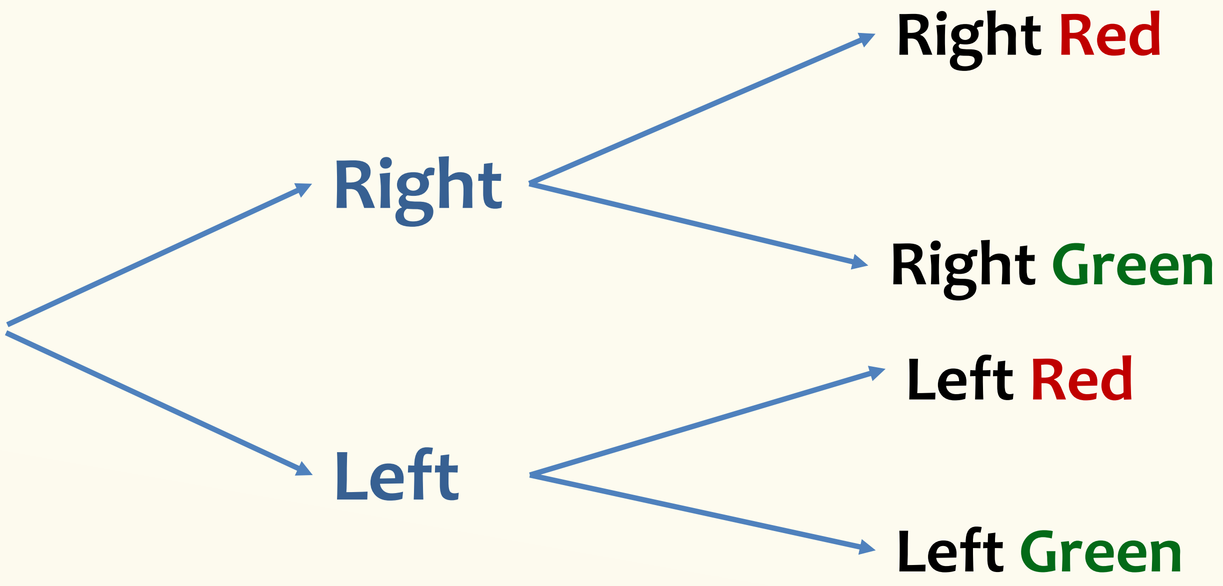
Alice has two pockets:

- **Left pocket:** Two red balls, two green balls
- **Right pocket:** One red ball, two green balls.

Alice picks a random ball from a random pocket.
[Both pockets equally likely, each ball equally likely.]

What is $\mathbb{P}(\mathbf{R})$ ?

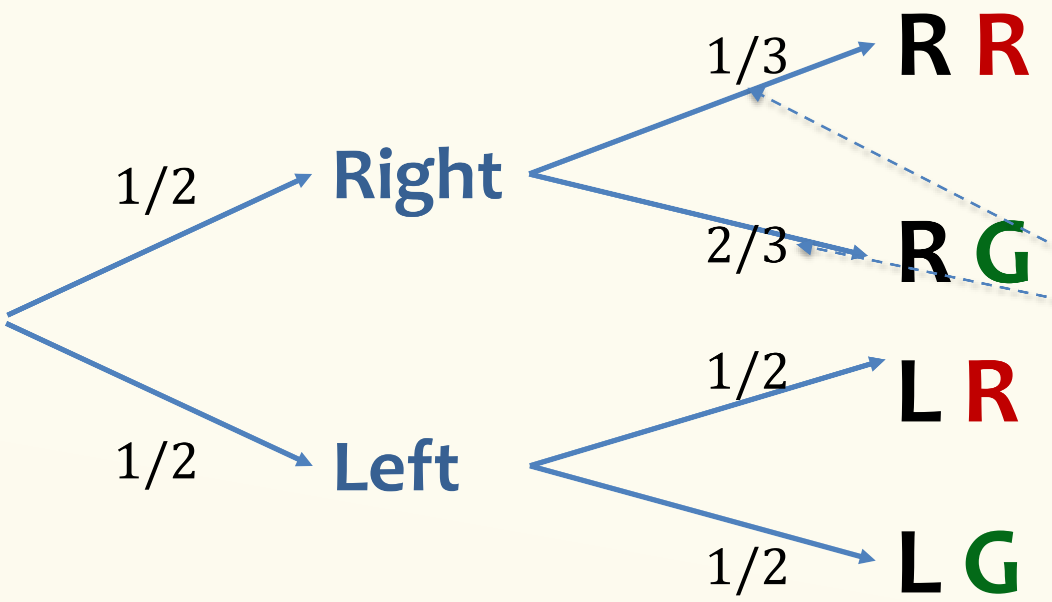
# Sequential Process – Non-Uniform Case




- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

- Alice picks a random ball from a random pocket

# Sequential Process – Non-Uniform Case




- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

1/3 = $\mathcal{P}(R \mid \mathrm{R})$ and 2/3 = $\mathcal{P}(G \mid \mathrm{R})$

$$\mathbb{P}(\mathbf{R}) = \mathbb{P}(\mathbf{R} \cap \mathbf{Left}) + \mathbb{P}(\mathbf{R} \cap \mathbf{Right})$$ **(Law of total probability)**

$$= \mathbb{P}(\mathbf{Left}) \times \mathbb{P}(\mathbf{R}|\mathbf{Left}) + \mathbb{P}(\mathbf{Right}) \times \mathbb{P}(\mathbf{R}|\mathbf{Right})$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

# Agenda

- Conditional Probability
- Time Permitting:
  - Bayes Theorem
  - Law of Total Probability
  - Bayes Theorem + Law of Total Probability
  - More Examples

# Our First Machine Learning Task: Spam Filtering

Subject: "FREE $$$ CLICK HERE"

What is the probability this email is spam, given the subject contains "FREE"?
Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word "FREE" in the subject.
- 70% of spam emails contain the word "FREE" in the subject.
- 80% of emails you receive are spam.

# Bayes Theorem with Law of Total Probability

**Bayes Theorem with LTP:** Let $E_1, E_2, \dots, E_n$ be a partition of the sample space, and $F$ and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

**Simple Partition:** In particular, if $E$ is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P(F|E^C)P(E^C)}$$

# Agenda

- Conditional Probability
- Time Permitting:
  - Bayes Theorem
  - Law of Total Probability
  - Bayes Theorem + Law of Total Probability
  - More Examples

# Example – Zika Testing


Usually no or mild symptoms (rash); sometimes severe symptoms (paralysis).

During pregnancy: may cause birth defects.

Suppose you took a Zika test, and it returns "positive", what is the likelihood that you actually have the disease?

- Tests for diseases are rarely 100% accurate.

This example and following slides are from Lisa Yan (Stanford).

# Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event $Z$) if you test positive (event $T$).

A) Less than 0.25
B) Between 0.25 and 0.5
C) Between 0.5 and 0.75
D) Between 0.75 and 1

# Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event $Z$) if you test positive (event $T$).

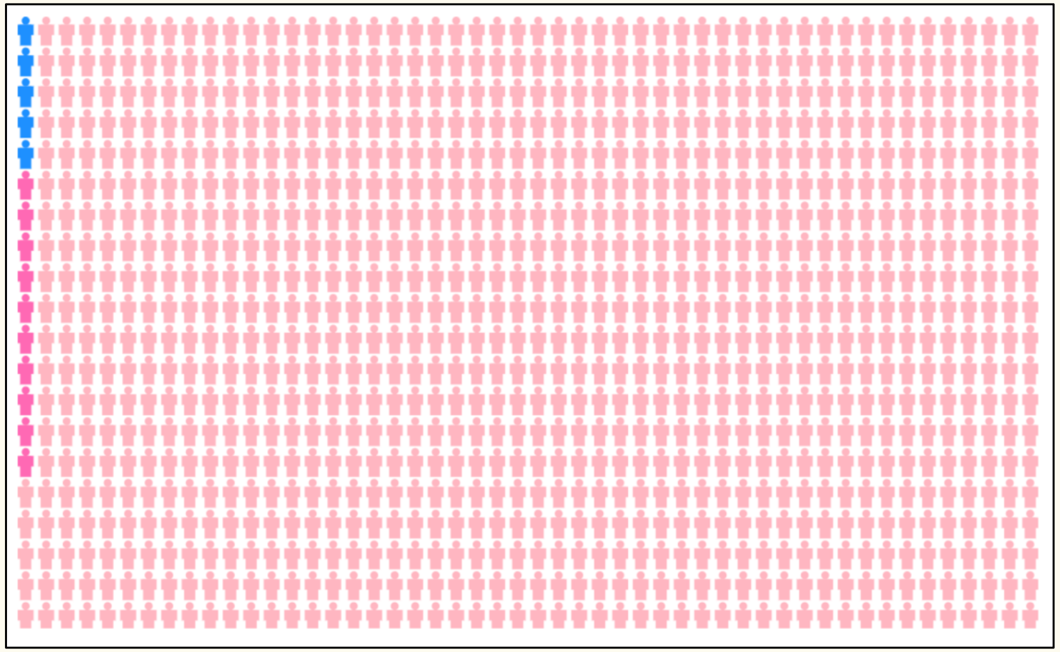
# Example – Zika Testing

Have zika blue, don't pink

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika. 5% have it.

What is the probability you have Zika (event *Z*) if you test positive (event *T*).



Suppose we had 1000 people:

- 5 **have Zika** and **test positive**
- 985 **do not have Zika** and **test negative**
- 10 **do not have Zika** and **test positive**

$$\frac{5}{5+10} = \frac{1}{3} \approx 0.33$$

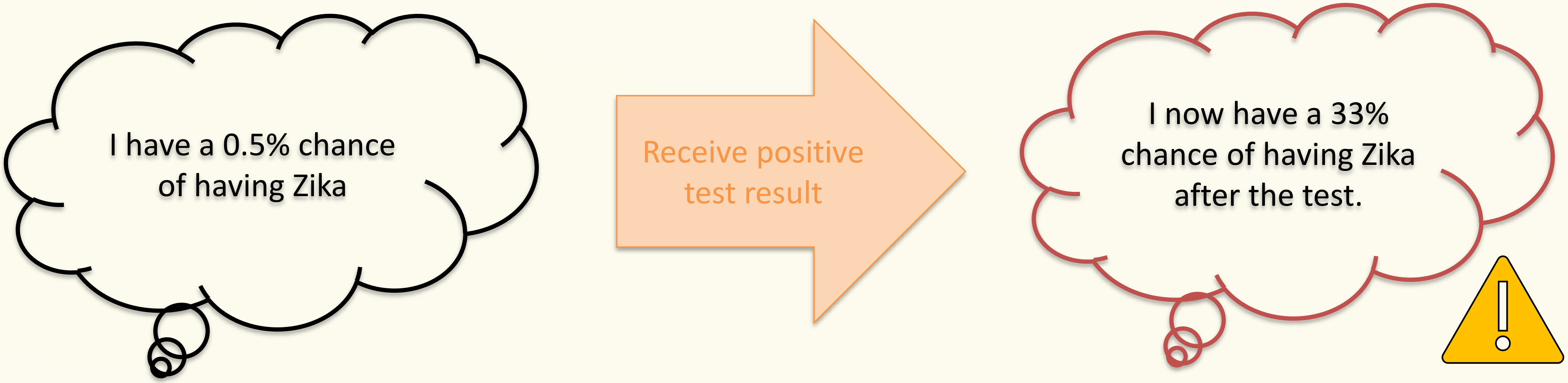
Demo

# Philosophy – Updating Beliefs

While it's not 98% that you have the disease, your beliefs changed **drastically**

Z = you have Zika

T = you test positive for Zika




**Prior:** $P(Z)$

**Posterior:** $P(Z|T)$

# Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you test negative (event $\bar{T}$) if you have Zika (event $Z$)?

# Conditional Probability Define a Probability Space

The probability conditioned on $A$ follows the same properties as (unconditional) probability.

**Example.** $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

# Conditional Probability Define a Probability Space

The probability conditioned on $A$ follows the same properties as (unconditional) probability.

**Example.** $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

**Formally.** $(\Omega, \mathbb{P})$ is a probability space + $\mathbb{P}(\mathcal{A}) > 0$

$\Rightarrow$ $(\mathcal{A}, \mathbb{P}(\cdot\,|\mathcal{A}))$ is a probability space