CSE 312

Foundations of Computing II

Lecture 4: Discrete probability

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺
Plus few slides from Berkeley CS 70

Probability

- We want to model uncertainty.
  - i.e., outcome not determined a-priori
  - E.g. throwing dice, flipping a coin…
  - We want to numerically measure likelihood of outcomes = probability.
  - We want to make complex statements about these likelihoods.
- First part of class: "Discrete" probability theory
  - Experiment with finite / discrete set of outcomes.
  - Will explore countably infinite and continuous outcomes later

Agenda

- Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- Examples

Sample Space

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

$\Omega = \{HH, HT, TH, TT\}$

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

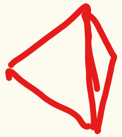
- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Definition. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$ (i.e., can't happen at same time)

Examples:

- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$

Example: 4-sided Dice



Suppose I roll two 4-sided dice Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. D1 = 1

B. D1 + D2 = 6

B = {(2,4) (3,3) (4,2)}

C. D1 = 2 * D2

Die 2 (D2)

Die 1 (D1)	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice

Suppose I roll two 4-sided dice Let D1 be the value of the blue die and D2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. D1 = 1

$$A = \{(1,1), (1,2), (1,3), (1,4)\}$$

B. D1 + D2 = 6

$$B = \{(2,4), (3,3), (4,2)\}$$

C. D1 = 2 * D2

$$C = \{(2,1), (4,2)\}$$

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice, Mutual Exclusivity

Are A and B mutually exclusive? YES
How about B and C? NO

A. D1 = 1

B. D1 + D2 = 6

C. D1 = 2 * D2

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
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Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

$$\mathbb{P}: \Omega \to [0, 1]$$

that maps outcomes $\omega \in \Omega$ to probabilities.

– Also use notation: $\mathbb{P}(\omega) = P(\omega) = \Pr(\omega)$

Example – Coin Tossing

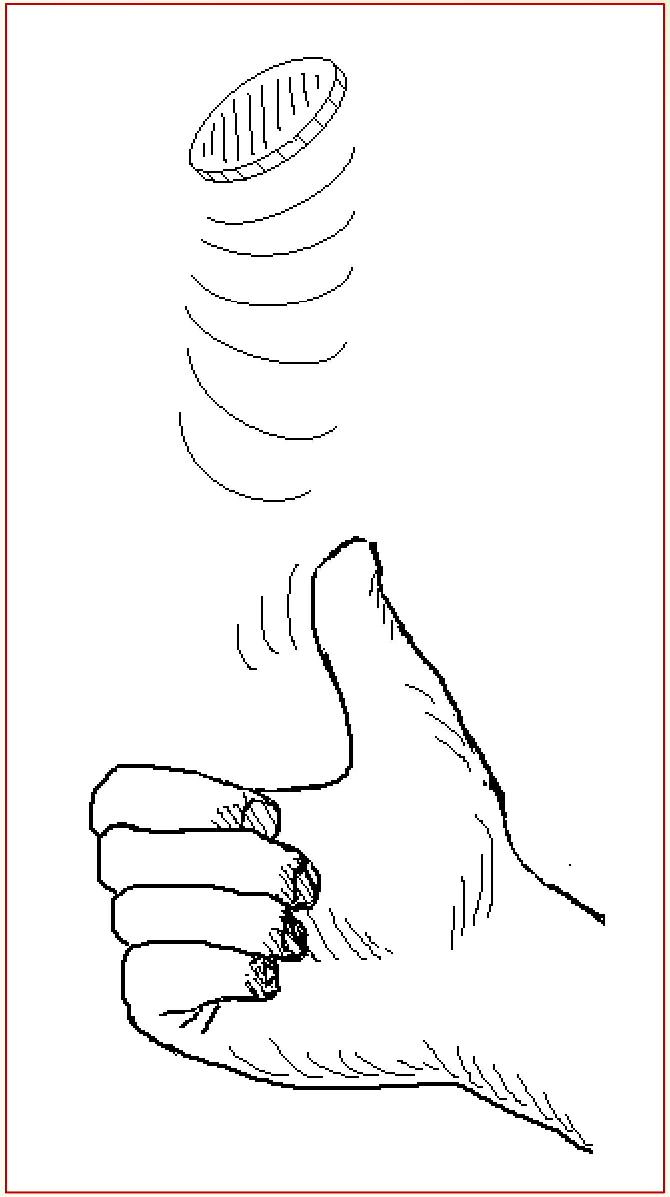
Imagine we toss one coin – outcome can be **heads** or **tails.**

$$\Omega = \{H, T\}$$

$\mathbb{P}$? Depends! What do we want to model?!

Fair coin toss

$$\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2} = 0.5$$



Example – Coin Tossing

Imagine we toss one coin – outcome can be **heads** or **tails.**

$$\Omega = \{\mathrm{H}, \mathrm{T}\}$$

$\mathbb{P}$? Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(\mathrm{H}) = 0.85, \qquad \mathbb{P}(\mathrm{T}) = 0.15$$

Probability space

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

- Ω is a set called the **sample space**.
- $\mathbb{P}$ is the **probability measure,** a function $\mathbb{P}: \Omega \to [0,1]$ such that:
  - $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$
  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Probability space

Either finite or infinite countable (e.g., integers)

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  - $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Set of possible **elementary outcomes**

Ω $\mathbb{P}$

H → 0.5

T → 0.5

Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Uniform Probability Space

Definition. A **uniform probability space** is a pair $(\Omega, \mathbb{P})$ such that

$$\mathbb{P}(\omega) = \frac{1}{|\Omega|}$$

for all $\omega \in \Omega$.

Examples:

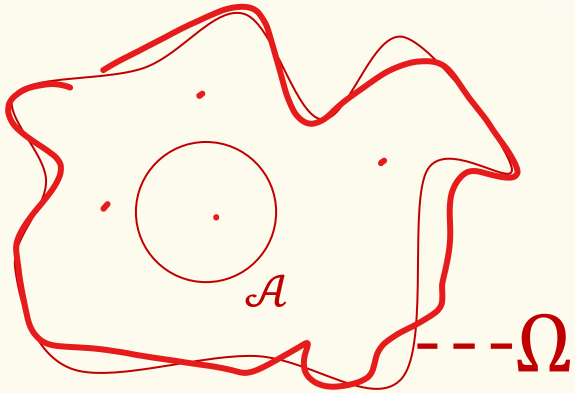
$\Omega = \{H, T\}$ $\quad \mathbb{P}(\omega) = \frac{1}{|\Omega|} = \frac{1}{2}$

- Fair coin $P(\omega) = \frac{1}{2}$
- Fair 6-sided die $P(\omega) = \frac{1}{6}$

Events

Definition. An **event** in a probability space $(\Omega, \mathbb{P})$ is a subset $\mathcal{A} \subseteq \Omega$. Its probability is

$$\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)$$



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Example: 4-sided Dice, Event Probability

Think back to 4-sided die. Suppose each die is fair. What is the probability of event B? $\Pr(B) = ???$

B. D1 + D2 = 6

$B = \{(2,4), (3,3)(4,2)\}$

Die 2 (D2)

Die 1 (D1)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
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Equally Likely Outcomes

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$, then

$$P(E) = \frac{|E|}{|\Omega|}$$

This follows from the definitions of the prob. of an event and uniform probability spaces.

Example – Coin Tossing

Toss a coin 100 times. Each outcome is **equally likely**. What is the probability of seeing 50 heads?

(A) $\frac{1}{2}$

(B) $\frac{1}{2^{50}}$

(C) $\frac{\binom{100}{50}}{2^{100}}$

(D) Not sure

E = seeing 50 heads

$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{\binom{100}{50}}{2^{100}}$

$2 \cdot 2 \cdot 2 \cdots = 2^{100}$

$\binom{100}{50}$

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Review Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair $(\Omega, \mathbb{P})$ where:

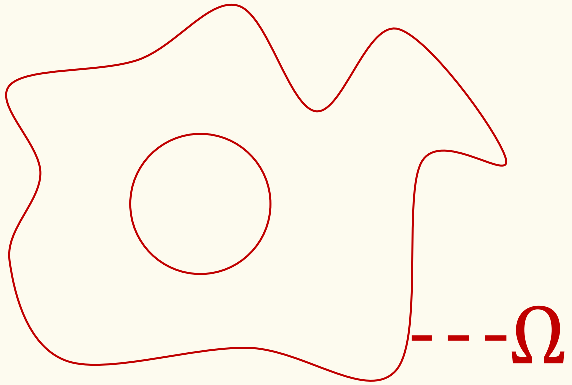
- Ω is a set called the **sample space.**
- $\mathbb{P}$ is the **probability measure,**

  a function $\mathbb{P}: \Omega \rightarrow [0,1]$ such that:

  – $\mathbb{P}(\omega) \geq 0$ for all $\omega \in \Omega$

  – $\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1$

Set of possible **elementary outcomes**



Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Axioms of Probability

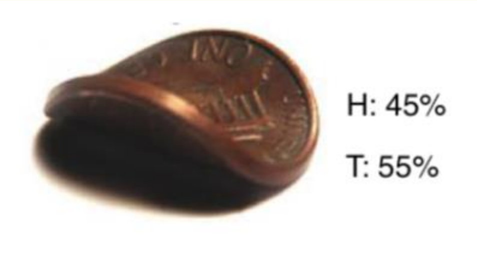
Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is applies to **any** probability space (not just uniform)

Axiom 1 (Non-negativity): $P(E) \geq 0.$
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

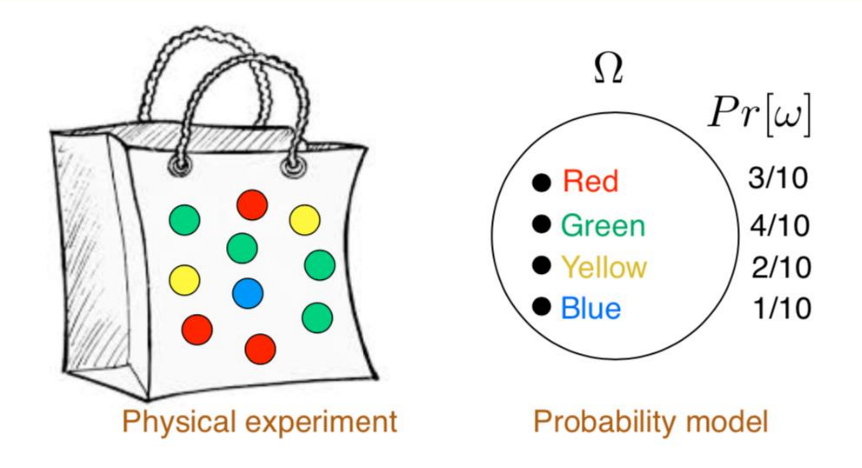
Corollary 1 (Complementation): $P(E^c) = 1 - P(E).$
~~**Corollary 2 (Monotonicity):** If $E \subseteq F$, $P(E) \leq P(F)$~~
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

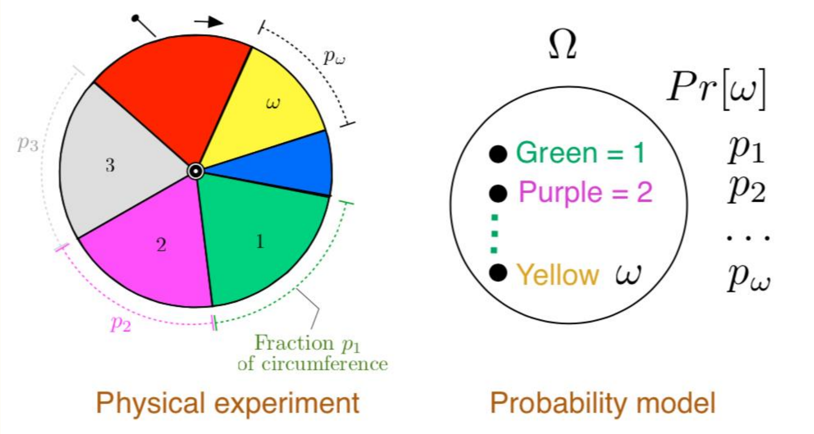
Non-equally Likely Outcomes

Probability spaces can have **non-equally likely outcomes.**



More Examples of Non-equally Likely Outcomes





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Example: Dice Rolls

Suppose I had a two, fair, 6-sided dice that we roll once each. What is the probability that we see *at least one 3 in the two rolls*.

There are a couple ways of doing this.

One way is to define event E as "seeing at least one 3 in the two rolls" and finding it directly. Do this by finding |E| = 11, |Omega| = 36, so the answer is 11/36.

We could also try to find P(A u B) where A is "seeing a 3 in the first roll" and B is "seeing a 3 in the second roll". Via inclusion-exclusion, this is P(A u B) = P(A) + P(B) - P(A n B). Via the same equally likely outcomes methods as before, we find P(A) = P(B) = 1/6. P(A n B) = 1/36. So, the final answer also becomes 11/36.

Example: Birthday "Paradox"

$\mathbb{P}$

$P(n, k)$

Suppose we have a collection of n people in a room. What is the probability that at least 2 people share a birthday? Assuming there are 365 possible birthdays, with uniform probability for each day.

$$\mathbb{P}(E) = 1 - \mathbb{P}(E^c) = 1 - \frac{|E^c|}{|\Omega|} = 1 - \frac{P(365, n)}{365^n}$$

$n = 57$

$n = 20\text{-}30$

Example: Birthday “Paradox” cont.

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