CSE 312
Foundations of Computing II

Lecture 4: Discrete probability

Slide Credit: Based on Stefano Tessaro’s slides for 312 19au incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself 😊
Plus few slides from Berkeley CS 70
Probability

• We want to model uncertainty.
  – i.e., outcome not determined a-priori
  – E.g. throwing dice, flipping a coin...
  – We want to numerically measure likelihood of outcomes = probability.
  – We want to make complex statements about these likelihoods.

• First part of class: “Discrete” probability theory
  – Experiment with finite / discrete set of outcomes.
  – Will explore countably infinite and continuous outcomes later
Agenda

• Events
• Probability
• Equally Likely Outcomes
• Probability Axioms and Beyond Equally Likely Outcomes
• Examples
Definition. A **sample space** $\Omega$ is the set of all possible outcomes of an experiment.

**Examples:**

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$
Events

Definition. An event $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:
• Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
• Rolling an even number on a die: $E = \{2, 4, 6\}$
Events

**Definition.** An event $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:
- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die: $E = \{2, 4, 6\}$

**Definition.** Events $E$ and $F$ are mutually exclusive if $E \cap F = \emptyset$ (i.e., can’t happen at the same time)

Examples:
- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$


Example: 4-sided Dice

Suppose I roll two 4-sided dice. Let $D_1$ be the value of the blue die and $D_2$ be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$

B. $D_1 + D_2 = 6$

C. $D_1 = 2 \times D_2$

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Die 1 ($D_1$)

Die 2 ($D_2$)
Example: 4-sided Dice

Suppose I roll two 4-sided dice. Let $D_1$ be the value of the blue die and $D_2$ be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$
   
   $A = \{(1,1), (1,2), (1,3), (1,4)\}$

B. $D_1 + D_2 = 6$
   
   $B = \{(2,4), (3,3), (4,2)\}$

C. $D_1 = 2 \times D_2$
   
   $C = \{(2,1), (4,2)\}$
Example: 4-sided Dice, Mutual Exclusivity

Are $A$ and $B$ mutually exclusive? **YES**
How about $B$ and $C$? **NO**

**A.** $D_1 = 1$

**B.** $D_1 + D_2 = 6$

**C.** $D_1 = 2 \times D_2$

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**Diagram:**

- **Die 1 ($D_1$):**
  - 1
  - 2
  - 3
  - 4

- **Die 2 ($D_2$):**
  - 1
  - 2
  - 3
  - 4

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Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

\[ \mathbb{P} : \Omega \rightarrow [0, 1] \]

that maps outcomes \( \omega \in \Omega \) to probabilities.

– Also use notation: \( \mathbb{P}(\omega) = P(\omega) = \Pr(\omega) \)
Example – Coin Tossing

Imagine we toss one coin – outcome can be heads or tails.

$$\Omega = \{H, T\}$$

$$\mathbb{P}$$? Depends! What do we want to model?!

**Fair coin toss**

$$\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2} = 0.5$$
Example – Coin Tossing

Imagine we toss one coin – outcome can be heads or tails.

$$\Omega = \{H, T\}$$

$$\mathbb{P}?$$ Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(H) = 0.85, \quad \mathbb{P}(T) = 0.15$$
Probability space

**Definition.** A (discrete) probability space is a pair \((\Omega, \mathbb{P})\) where:

- \(\Omega\) is a set called the **sample space**.
- \(\mathbb{P}\) is the **probability measure**, a function \(\mathbb{P}: \Omega \rightarrow [0,1]\) such that:
  - \(\mathbb{P}(\omega) \geq 0\) for all \(\omega \in \Omega\)
  - \(\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1\)
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  - \(\sum_{\omega \in \Omega} \mathbb{P}(\omega) = 1\)

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Either finite or infinite countable (e.g., integers)

Set of possible elementary outcomes

Specify Likelihood (or probability) of each elementary outcome
**Uniform Probability Space**

**Definition.** A **uniform probability space** is a pair $(\Omega, \mathbb{P})$ such that

$$
\mathbb{P}(\omega) = \frac{1}{|\Omega|}
$$

for all $\omega \in \Omega$.

**Examples:**
- Fair coin $P(\omega) = \frac{1}{2}$
- Fair 6-sided die $P(\omega) = \frac{1}{6}$
Definition. An event in a probability space \((\Omega, \mathbb{P})\) is a subset \(\mathcal{A} \subseteq \Omega\). Its probability is

\[
\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)
\]
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Think back to 4-sided die. Suppose each die is fair. What is the probability of event $B$? $\Pr(B) = \square$?

**Example: 4-sided Dice, Event Probability**

**B.** $D_1 + D_2 = 6$

$B = \{(2,4), (3,3), (4,2)\}$

$$\Pr(B) = \sum_{w \in B} \Pr(w) = \Pr((2,4)) + \Pr((3,3)) + \Pr((4,2)) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{3}{16}$$
Equally Likely Outcomes

If $(\Omega, P)$ is a **uniform** probability space, then for any event $E \subseteq \Omega$, then

$$P(E) = \frac{|E|}{|\Omega|}$$

This follows from the definitions of the prob. of an event and uniform probability spaces.
Example – Coin Tossing

Toss a coin 100 times. Each outcome is **equally likely**. What is the probability of seeing 50 heads?

\[
P(E) = \frac{\binom{100}{50}}{2^{100}}
\]

(A) \(\frac{1}{2}\)

(B) \(\frac{1}{2^{50}}\)

(C) \(\frac{100}{50} \cdot \frac{50}{2^{100}}\)

(D) Not sure
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**Definition.** A (discrete) **probability space** is a pair \((\Omega, \mathbb{P})\) where:

- \(\Omega\) is a set called the **sample space**.
- \(\mathbb{P}\) is the **probability measure**, a function \(\mathbb{P} : \Omega \rightarrow [0,1]\) such that:
  
  \[
  \begin{align*}
  - \mathbb{P}(\omega) &\geq 0 \text{ for all } \omega \in \Omega \\
  - \sum_{\omega \in \Omega} \mathbb{P}(\omega) &\leq 1
  \end{align*}
  \]

- Either finite or infinite countable (e.g., integers)

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Set of possible elementary outcomes

Specify Likelihood (or probability) of each elementary outcome
Axioms of Probability

Let $\Omega$ denote the sample space and $E, F \subseteq \Omega$ be events. Note this is applies to any probability space (not just uniform)

**Axiom 1 (Non-negativity):** $P(E) \geq 0$.

**Axiom 2 (Normalization):** $P(\Omega) = 1$.

**Axiom 3 (Countable Additivity):** If $E$ and $F$ are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$.

**Corollary 1 (Complementation):** $P(E^c) = 1 - P(E)$.

**Corollary 2 (Monotonicity):** If $E \subseteq F$, $P(E) \leq P(F)$.

**Corollary 3 (Inclusion-Exclusion):** $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.
Non-equally Likely Outcomes

Probability spaces can have non-equally likely outcomes.
More Examples of Non-equally Likely Outcomes

- Physical experiment
- Probability model

\[ \Omega \]

\[ Pr[\omega] \]

- Red: 3/10
- Green: 4/10
- Yellow: 2/10
- Blue: 1/10

- Physical experiment
- Probability model

\[ \Omega \]

\[ Pr[\omega] \]

- Green = 1
- Purple = 2
- Yellow

Fraction of circumference
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Example: Dice Rolls

Suppose I had a two, fair, 6-sided dice that we roll once each. What is the probability that we see \textit{at least one 3 in the two rolls}.

There are a couple ways of doing this.

One way is to define event $E$ as "seeing at least one 3 in the two rolls" and finding it directly. Do this by finding $|E| = 11$, $|\Omega| = 36$, so the answer is $11/36$.

We could also try to find $P(A \cup B)$ where $A$ is "seeing a 3 in the first roll" and $B$ is "seeing a 3 in the second roll". Via inclusion-exclusion, this is $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Via the same equally likely outcomes methods as before, we find $P(A) = P(B) = 1/6$. $P(A \cap B) = 1/36$. So, the final answer also becomes $11/36$. 
Example: Birthday “Paradox”

Suppose we have a collection of \( n \) people in a room. What is the probability that at least 2 people share a birthday? Assuming there are 365 possible birthdays, with uniform probability for each day.

\[
\Pr(E) = 1 - \Pr(E^c) = 1 - \frac{1^{365}}{365^n} = 1 - \frac{P(365, n)}{365^n}
\]

\( n = 57 \)
\( n = 70-30 \)
Example: Birthday “Paradox” cont.
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