CSE 312

Foundations of Computing II

Lecture 3: Counting III

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Agenda

- Recap
- Inclusion-Exclusion
- Binomial Theorem
- Pigeonhole Principle
- Combinatorial Proofs

$\binom{n}{k}$

Recap

- Permutations

$$P(n,k) = \frac{n!}{(n-k)!}$$

- Combinations

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

Recap

- Multinomial Coefficients (bonus content in textbook, 1.2)

- Stars and Bars

Ways to distribute n indistinguishable balls into k distinct bins =

$$\binom{n+k-1}{k-1}$$

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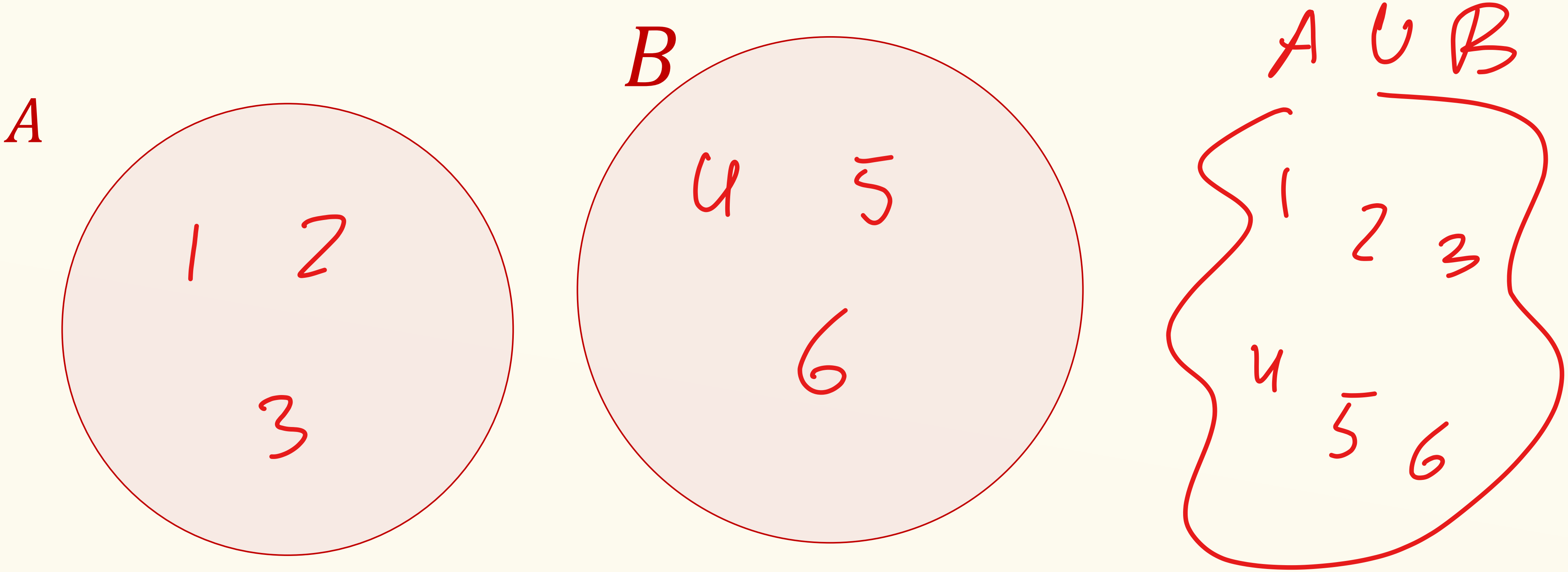
Recap Disjoint Sets

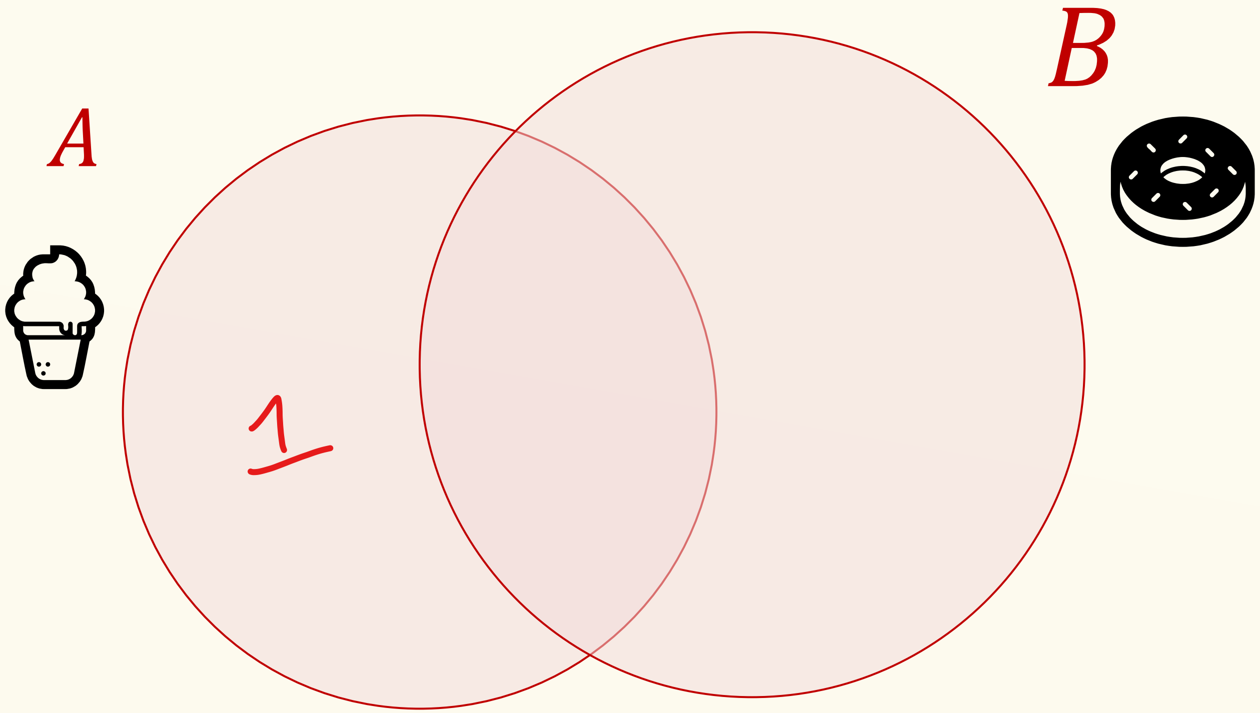
Sets that do not contain common elements ($A \cap B = \emptyset$)



Sum Rule: $|A \cup B| = |A| + |B|$

Inclusion-Exclusion

But what if the sets are not disjoint?

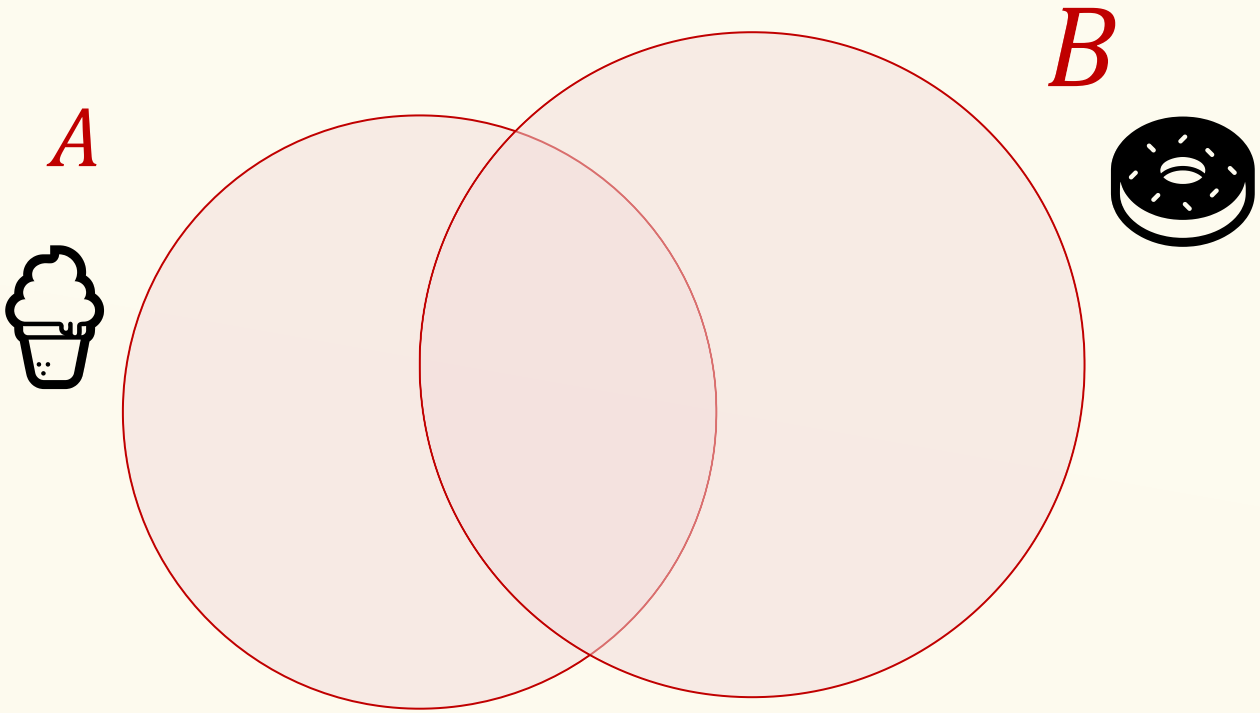


$|A| = 43$
$|B| = 20$
$|A \cap B| = 7$
$|A \cup B| = ???$

43

Inclusion-Exclusion

But what if the sets are not disjoint?



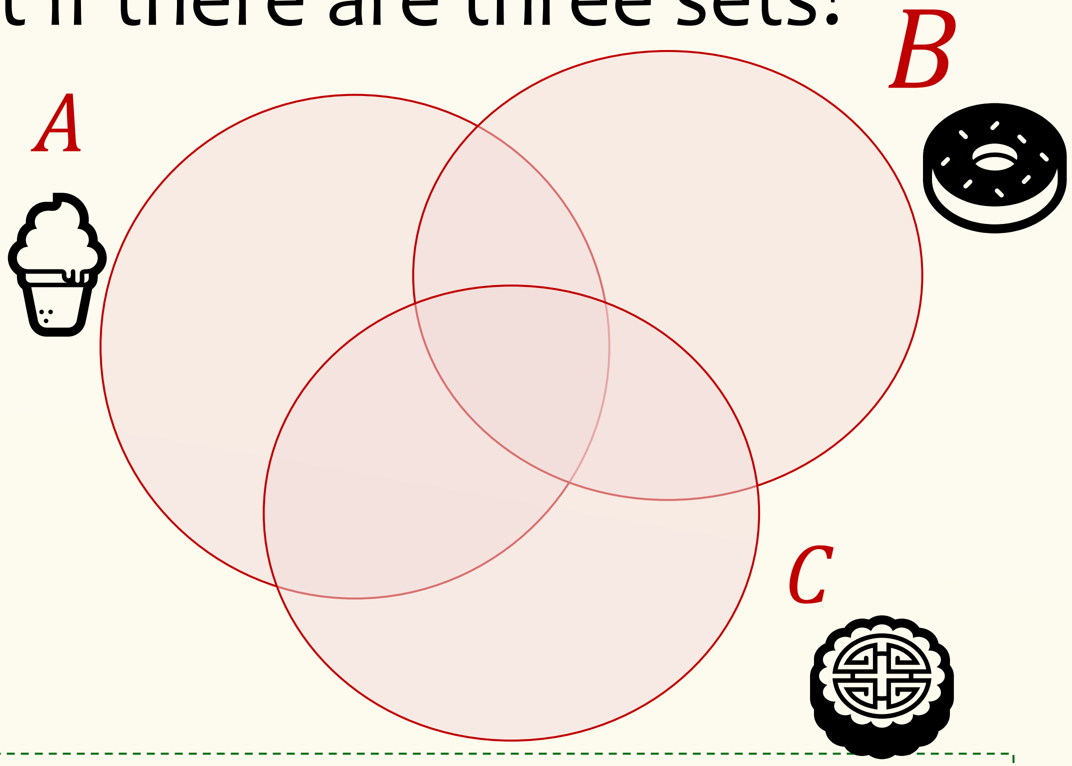
$|A| = 43$
$|B| = 20$
$|A \cap B| = 7$
$|A \cup B| = ???$

Fact. $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion

Not drawn to scale

What if there are three sets?



$|A| = 43$
$|B| = 20$
$|C| = 35$
$|A \cap B| = 7$
$|A \cap C| = 16$
$|B \cap C| = 11$
$|A \cap B \cap C| = 4$
$|A \cup B \cup C| = ???$

Fact.

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion-Exclusion

Let A, B be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$\begin{aligned} |A_1 \cup A_2 \cup \cdots \cup A_n| &= singles\ - doubles + triples\ - quads + \ \ldots \\ &= (|A_1| + \cdots + |A_n|)\ - (|A_1 \cap A_2| + \ \ldots + |A_{n-1} \cap A_n|) + \ \ldots \end{aligned}$$

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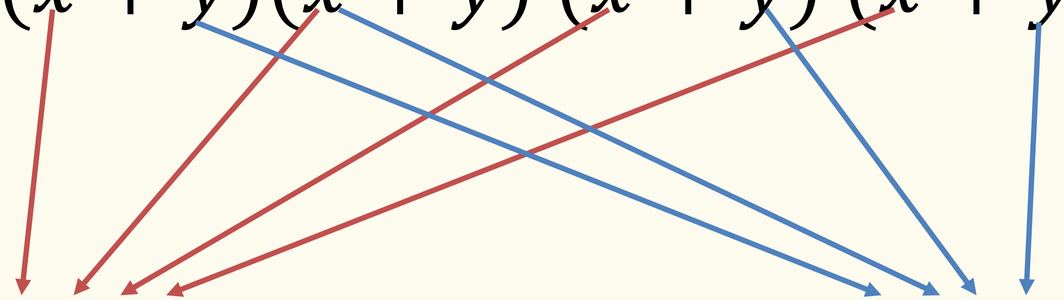
Binomial Theorem: Idea

$$(x+y)^2 = (x+y)(x+y)$$
$$= xx + xy + yx + yy$$
$$= x^2 + 2xy + y^2$$

$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$



$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Binomial Theorem: Idea

Poll: What is the coefficient for xy^3?

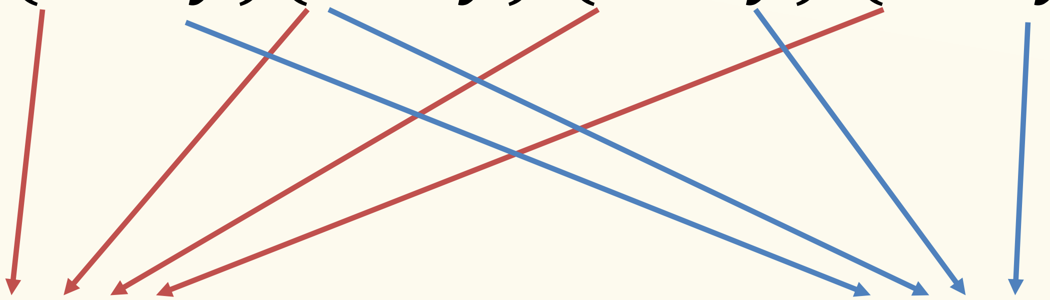
A. 4

B. $\binom{4}{1}$

C. $\binom{4}{3}$

D. 3

$$(x+y)^4 = (x+y)(x+y)\,(x+y)\,(x+y)$$



$$= xxxx + yyyy + xyxy + yxyy + \ \ldots$$

Binomial Theorem: Idea

$$(x + y)^n = (x + y)(x + y)(x + y) \cdots (x + y)$$

Each term is of the form $x^k y^{n-k}$, since each term is is made by multiplying exactly n variables, either x or y.

How many times do we get $x^k y^{n-k}$? The number of ways to choose k of the n variables we multiple to be an x (the rest will be y).

$$\binom{n}{k} = \binom{n}{n-k}$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

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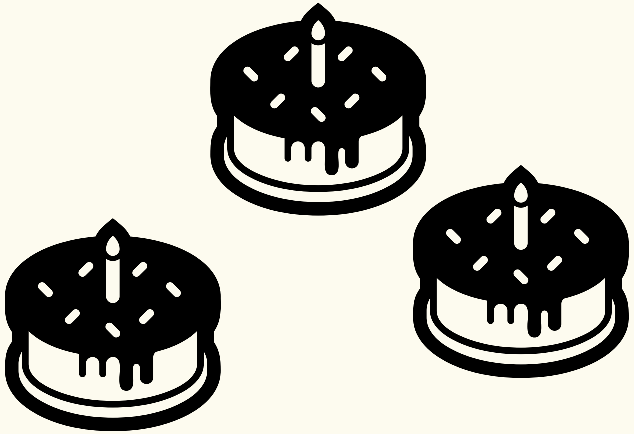
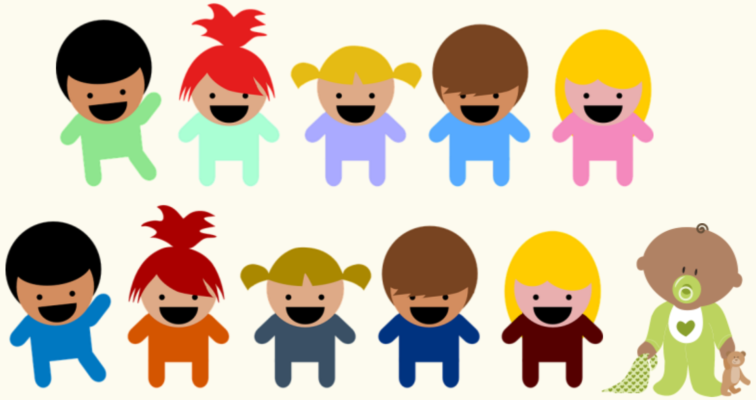
Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes

Pigeonhole Principle: Idea





If 11 children have to share 3 cakes, at least one cake must be shared by how many children?

Pigeonhole Principle – More generally

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\frac{n}{k}$ pigeons!

Proof. Assume there are $< \frac{n}{k}$ pigeons per hole.

Then, there are $< k\frac{n}{k} = n$ pigeons overall.

Contradiction!

Pigeonhole Principle – Better version

If there are n pigeons in $k < n$ holes, then one hole must contain at least $\left\lceil \frac{n}{k} \right\rceil$ pigeons!

Reason. Can't have fractional number of pigeons

Syntax reminder:

- Ceiling: $\lceil x \rceil$ is x rounded up to the nearest integer (e.g., $\lceil 2.731 \rceil = 3$)
- Floor: $\lfloor x \rfloor$ is x rounded down to the nearest integer (e.g., $\lfloor 2.731 \rfloor = 2$)

Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

Solution:

1. **367** pigeons = people
2. **365** holes = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday

Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeonhole Principle – Example (Surprising?)

In every set S of 100 integers, there are at least ***two*** *elements whose difference is a multiple of 37.*

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

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Combinatorial proof: Show that $M = N$

- Let S be a set of objects
- Show how to count $|S|$ one way => $|S| = M$
- Show how to count $|S|$ another way => $|S| = N$
- Conclude that $M = N$

Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$ **Symmetry in Binomial Coefficients**

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ **Pascal's Identity**

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ **Follows from Binomial theorem**

Pascal's Identities

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ \ldots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

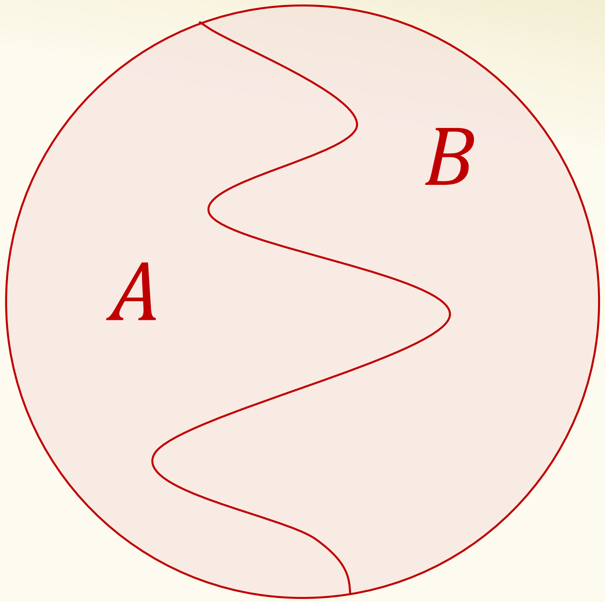
Hard work and not intuitive

Let's see a combinatorial argument

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \quad = \quad |A| \quad + \quad |B|$



$S = A \cup B$, disjoint

S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

A: the set of size k subsets of $[n]$ including n

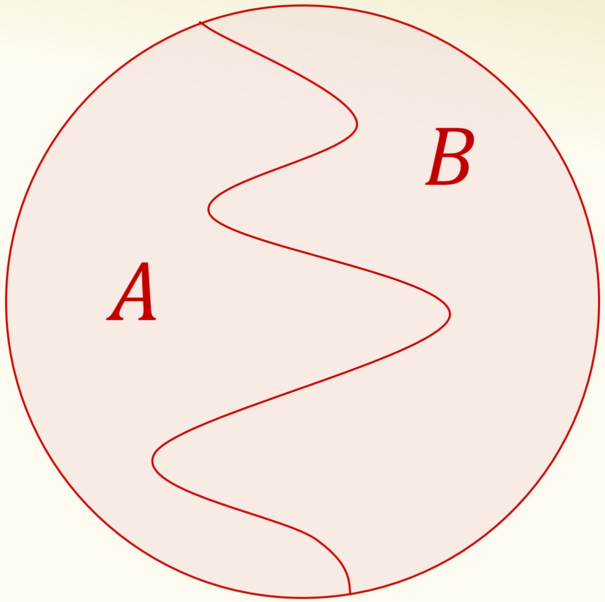
B: the set of size k subsets of $[n]$ NOT including n

Sum rule:
$|A \cup B| = |A| + |B|$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

e.g.: $n = 4, k = 2, \quad S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: the set of size 2 subsets of [4] including 4

$$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}.$$

B: the set of size 2 subsets of [4] NOT including 4

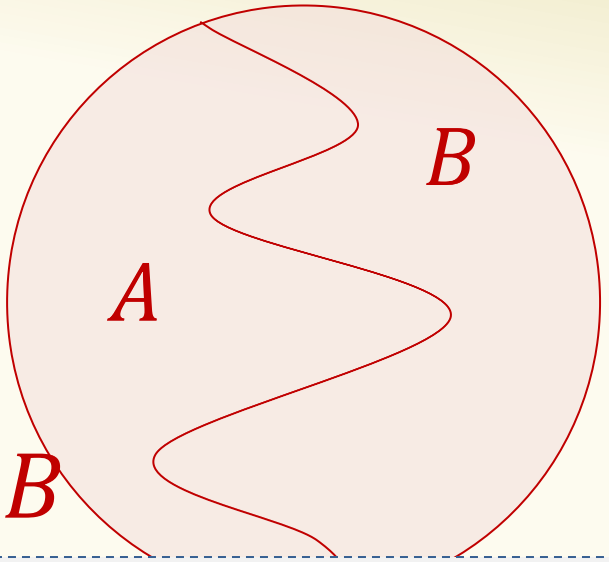
$$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S|$ $\quad |A|$ $\quad |B|$

$S = A \cup B$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$

A: the set of size k subsets of $[n]$ including n

n is in set, need to choose $k - 1$ elements from $[n - 1]$

$$|A| = \binom{n-1}{k-1}$$

B: the set of size k subsets of $[n]$ NOT including n

n not in set, need to choose k elements from $[n - 1]$

$$|B| = \binom{n-1}{k}$$

combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive

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Counting Recap

- **Core Theorems**
  - Sum & Product Rule
  - Permutations and Combinations
  - Inclusion-Exclusion
  - Binomial Theorem
- **Counting Strategies**
  - Complimentary Counting
  - Stars and Bars
  - Pigeonhole Principle
  - Combinatorial Proofs