CSE 312

Foundations of Computing II

Lecture 2: Counting II

1.2

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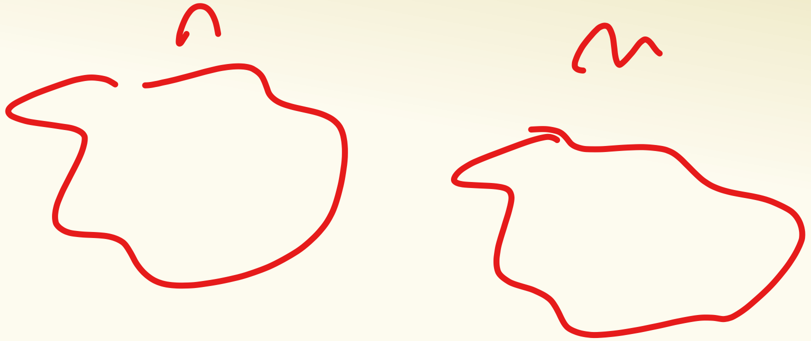
Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Anna Karlin, Alex Tsun, Rachel Lin, Hunter Schafer & myself ☺

Agenda

- Recap
- k-Permutations
- Combinations
- Multinomial Coefficients
- Stars and Bars
- How to Answer a Question

Quick Summary



- **Sum Rule**

  If you can choose from
  - Either one of n options,
  - OR one of m options with NO overlap with the previous n,

  then the number of possible outcomes of the experiment is $n + m$

- **Product Rule**

  In a sequential process, if there are
  - n_1 choices for the first step,
  - n_2 choices for the second step (given the first choice), …, and
  - n_k choices for the k^{th} step (given the previous choices),

  then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_k$

Quick Summary

$n \cdot (n-1) \cdot \ldots \cdot 1 = n!$

- Permutations: How many ways to order n distinct items?
  - Product rule → n!

- Complementary Counting: Instead of counting |S|, count |U| - |U/S|

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Distinct Letters

$26 \cdot 26 \cdot 26 \cdot 26 \cdot 26$

"How many sequences of 5 distinct alphabet letters from $\{A, B, \ldots, Z\}$*?"*

E.g., AZURE, BINGO, TANGO. But not: STEVE, SARAH

ABCDE

$26 \times 25 \times 24 \times 23 \times 22$

A…Z A…Z

this one

Distinct Letters

"How many sequences of 5 *distinct alphabet letters from* $\{A, B, \dots, Z\}$*?"*

E.g., AZURE, BINGO, TANGO. But not: STEVE, SARAH

Answer: $26 \times 25 \times 24 \times 23 \times 22 = 7893600$

In general

Aka: k-permutations

Fact. # of ways to arrange k out of n distinct objects in a sequence.

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

We say ``n pick k''

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Number of Subsets

"How many size-5 ***subsets*** *of* $\{A, B, \dots, Z\}$*?"*

E.g., {A,Z,U,R,E}, {B,I,N,G,O}, {T,A,N,G,O}. But not:
{S,T,E,V}, {S,A,R,H},…

Number of Subsets

"How many size-5 ***subsets*** *of* $\{A, B, \dots, Z\}$*?"*

E.g., {A,Z,U,R,E}, {B,I,N,G,O}, {T,A,N,G,O}. But not: {S,T,E,V}, {S,A,R,H},…

Difference from k-permutations: **NO ORDER**

Different sequences: TANGO, OGNAT, ATNGO, NATGO, ONATG …

Same set: {T,A,N,G,O}, {O,G,N,A,T}, {A,T,N,G,O}, {N,A,T,G,O}, {O,N,A,T,G}… …

How to count number of 5 element subsets of $\{A, B, \dots, Z\}$?

Consider the following process:

1. Choose an **unordered** subset $S \subseteq \{A, B, \dots, Z\}$ of size $|S| = 5$
   e.g. $S = \{A, G, N, O, T\}$
2. Choose a permutation of letters in S
   e.g., *TANGO, AGNOT, NAGOT, GOTAN, GOATN, NGOAT, …*

Outcome: An **ordered** sequence of 5 distinct letters from $\{A, B, \dots, Z\}$

$? \times 5! = \frac{26!}{21!}$

$? = \frac{26!}{21!\,5!}$

Number of Subsets – Idea for how to count

Consider the following process:

1. Choose an **unordered** subset $S \subseteq \{A, B, \dots, Z\}$ of size $|S| = 5$. e.g. $S = \{A, G, N, O, T\}$

1. Choose a permutation of letters in S
   e.g., *TANGO, AGNOT, NAGOT, GOTAN, GOATN, NGOAT, …*

Outcome: An **ordered** sequence of 5 distinct letters from $\{A, B, \dots, Z\}$

$$??? \times 5! = \frac{26!}{21!}$$

$$??? = \frac{26!}{21!\,5!} = 65780$$

Combinations

$\frac{n!}{(n-k)!} = P(n,k)$

Fact. The number of subsets of size k of a set of size n is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$$

{A, C, G, F, B}

$26 = n$

$5 = k$

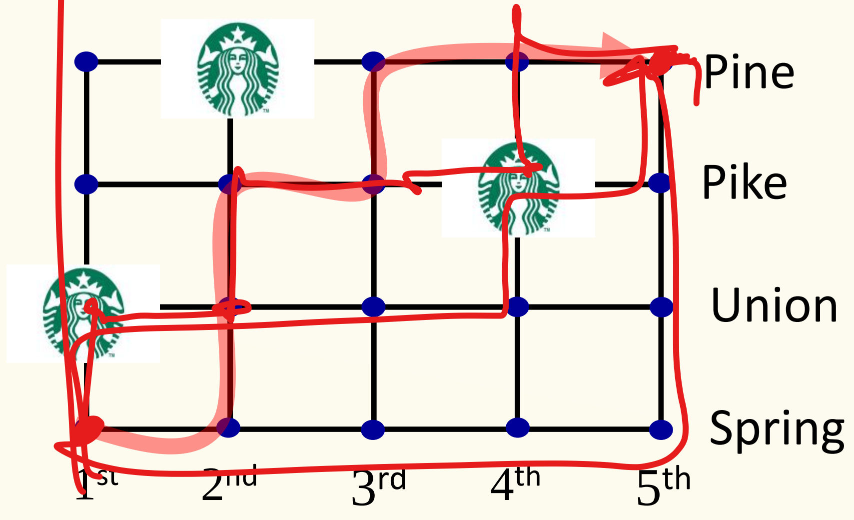
we say "n choose k"

[also called **binomial coefficients**]

$\frac{26!}{21!\,5!} \Rightarrow \frac{n!}{(n-k)!\,k!}$

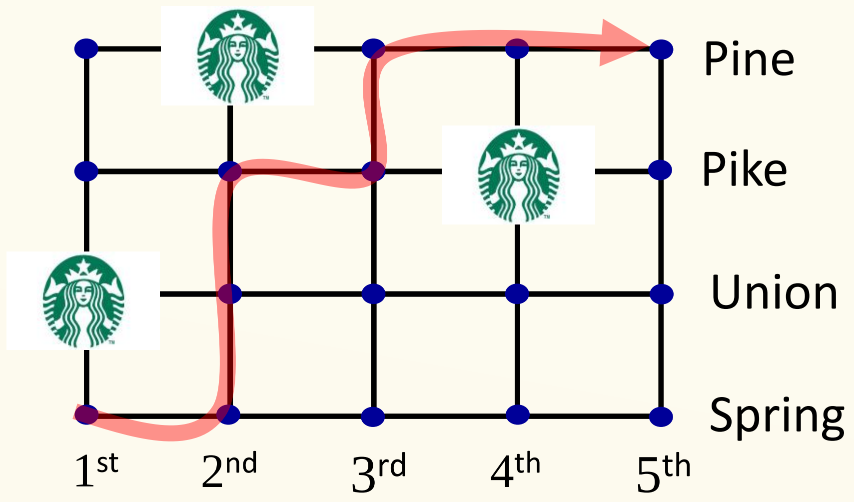
$P(n,k)$ sequences vs subsets $\binom{n}{k}$

Example – Counting Paths



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → ?"

Example – Counting Paths -2



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → ?

Poll:

A. 2^7

B. $\frac{7!}{4!}$ = P(7, 3)

C. $\binom{7}{4} = \frac{7!}{4!3!}$

D. $\binom{7}{3} = \frac{7!}{3!4!}$

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

Why??

This is called an Algebraic proof,
i.e., Prove by checking algebra

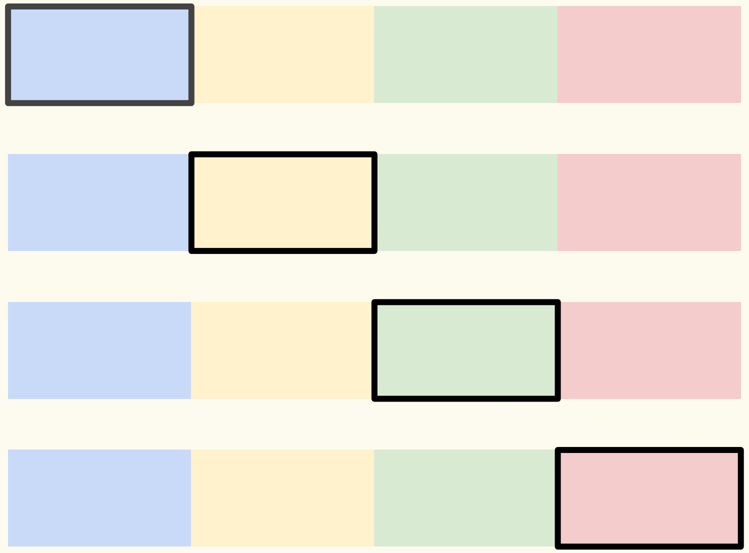
Symmetry in Binomial Coefficients – A different proof

Fact. $\binom{n}{k} = \binom{n}{n-k}$

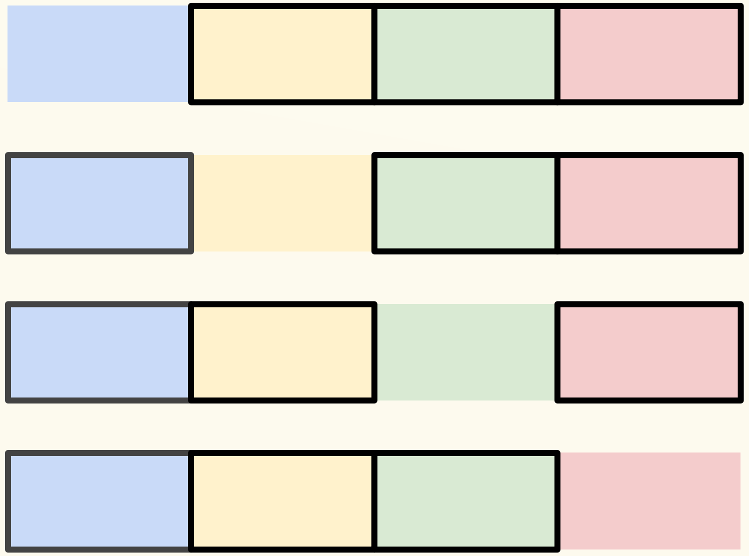
$n = 4$
$k = 1$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n-k$ elements are excluded



$$\binom{4}{1} = 4 = \binom{4}{3}$$



Symmetry in Binomial Coefficients – A different proof

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

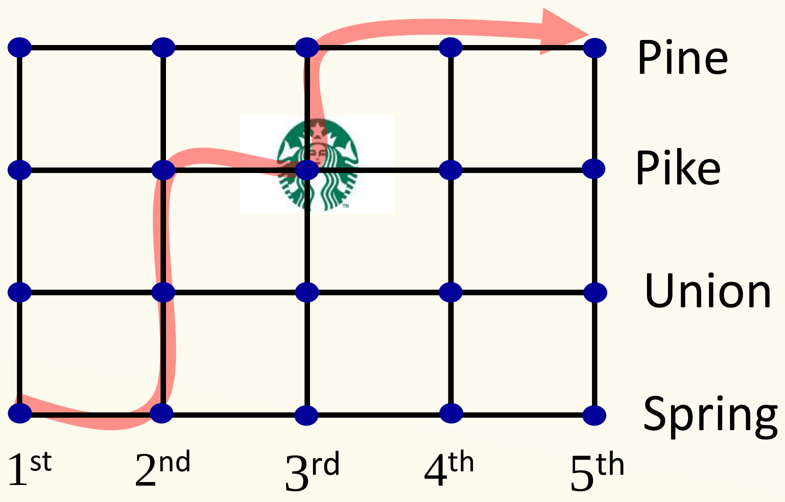
1. Choose which k elements are included
2. Choose which $n - k$ elements are excluded

This is called a combinatorial argument/proof

- Let S be a set of objects
- Show how to count $|S|$ one way => $|S| = N$
- Show how to count $|S|$ another way => $|S| = m$

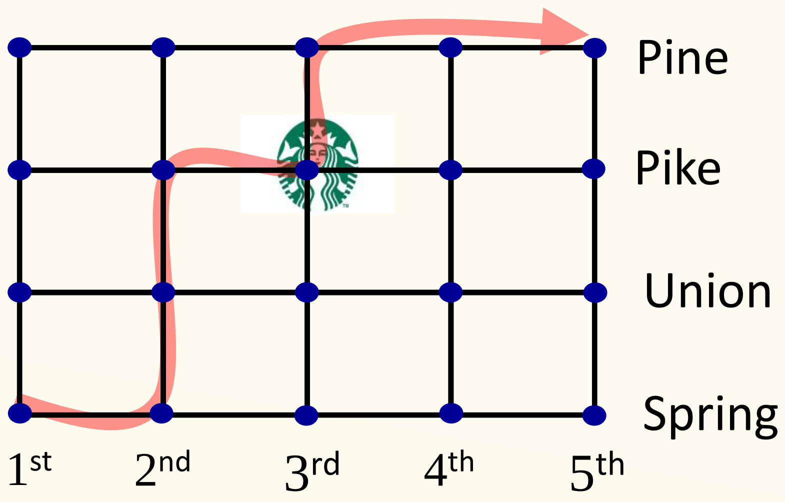
More examples of combinatorial proofs coming soon!

Example – Counting Paths - 3



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → but stopping at Starbucks on 3rd and Pike?"

Example – Counting Paths - 3



"How many ways to walk from 1st and Spring to 5th and Pine only going ↑ and → but stopping at Starbucks on 3rd and Pike?"

<u>Poll:</u>

A. $\binom{7}{3}\binom{7}{3}\binom{7}{1}$

B. $\binom{4}{2}\binom{3}{1}$

C. $\binom{4}{2}\binom{3}{2}$

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Example – Word Permutations

How many ways to re-arrange the letters in the word "MATH"?

<u>Poll:</u>

A. $\binom{26}{4}$

B. 4^4

C. $4!$

D. *I don't know*

MATH

Example – Word Permutations

How many ways to re-arrange the letters in the word "MUUMUU"?

Example – Word Permutations

How many ways to re-arrange the letters in the word "MUUMUU"?

Choose where the 2 M's go, and then the U's are set **OR**
Choose where the 4 U's go, and then the M's are set

Either way, we get $\binom{6}{2} \cdot \binom{4}{4} = \binom{6}{4} \cdot \binom{2}{2} = \frac{6!}{2!4!}$

Another way to think about it

How many ways to re-arrange the letters in the word "MUUMUU"?

Arrange the 6 letters as if they were distinct.

$$M_1U_1U_2M_2U_3U_4$$

Then divide by 4! to account for duplicate M's and divide by 2! to account for duplicate U's.

Yields $\frac{6!}{2!4!}$

Example – Word Permutations

How many ways to re-arrange the letters in the word "GODOGGY"?

Poll:

A. $7!$

B. $\frac{7!}{3!}$

C. $\frac{7!}{3!2!1!1!}$

D. $\binom{7}{3} \cdot \binom{5}{2} \cdot 3!$

Example – Word Permutations

How many ways to re-arrange the letters in the word "GODOGGY"?

n= 7 (length of sequence)   K = 4 types = {G, O, D, Y}

$n_1 = 3, n_2 = 2, n_3 = 1, n_4 = 1$

$$\binom{7}{3,2,1,1} = \frac{7!}{3!2!1!1!}$$

Multinomial coefficients

If we have k types of objects, with n_1 of the first type, n_2 of the second type, … , n_k of the kth type, where $n = n_1 + n_2 + \cdots + n_k$ then the number of arrangements of the n objects is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!\, n_2! \cdots n_k\,!}$$

Note that objects of the same type are indistinguishable.

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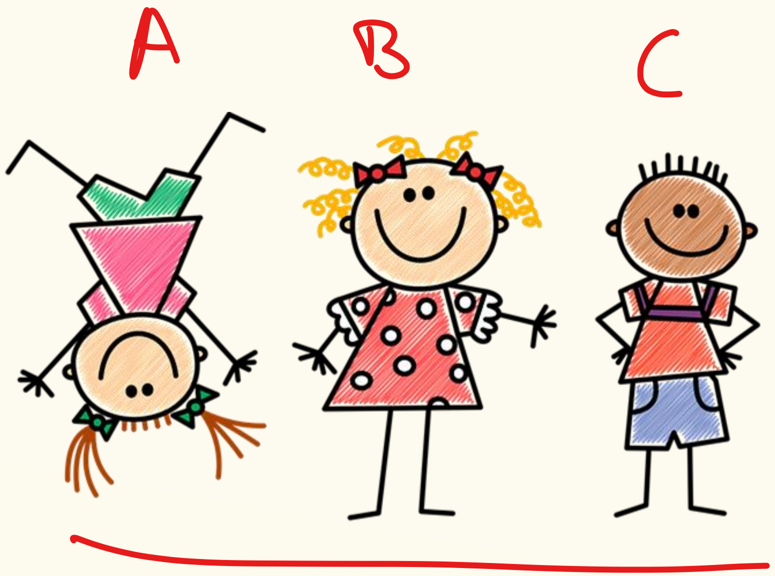
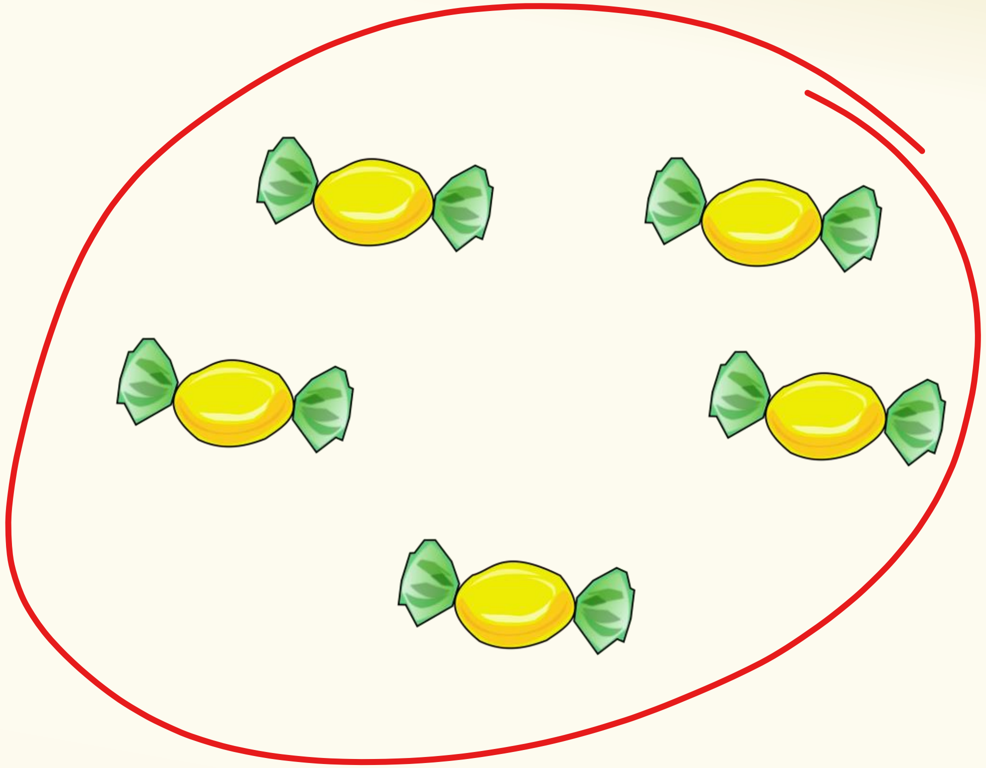
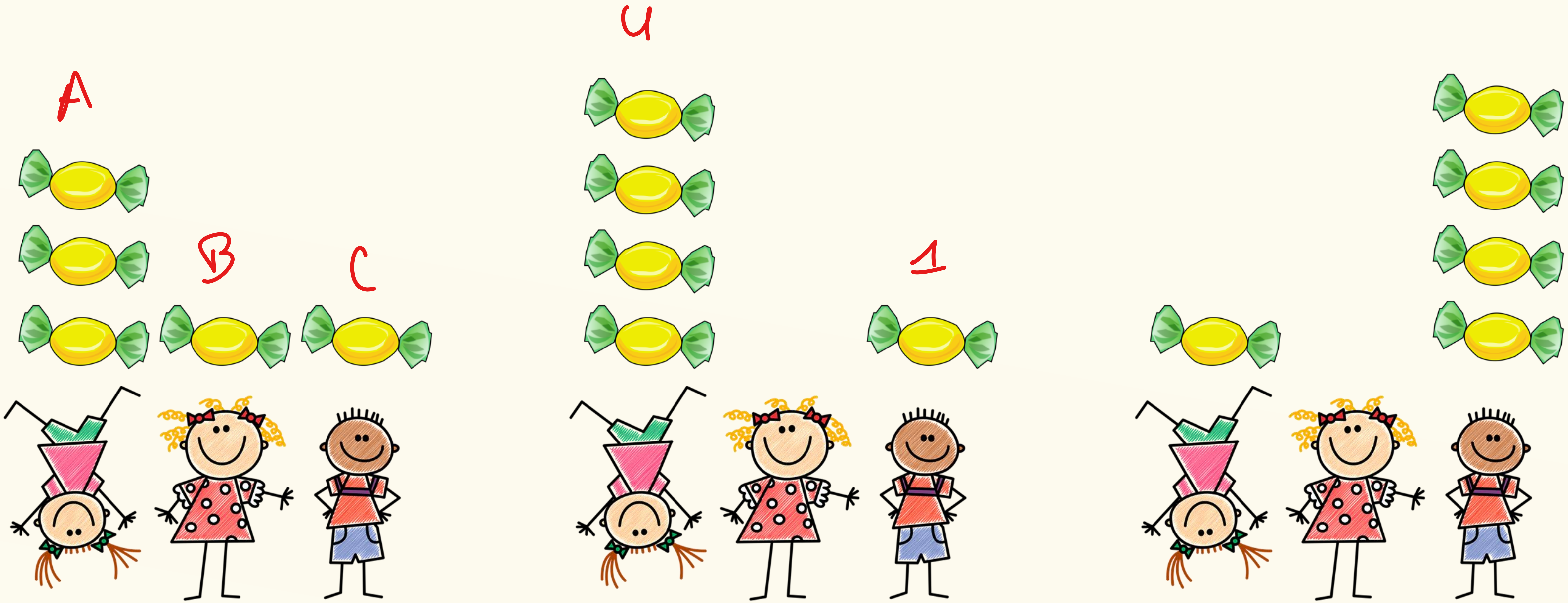
Example: Kids and Candies



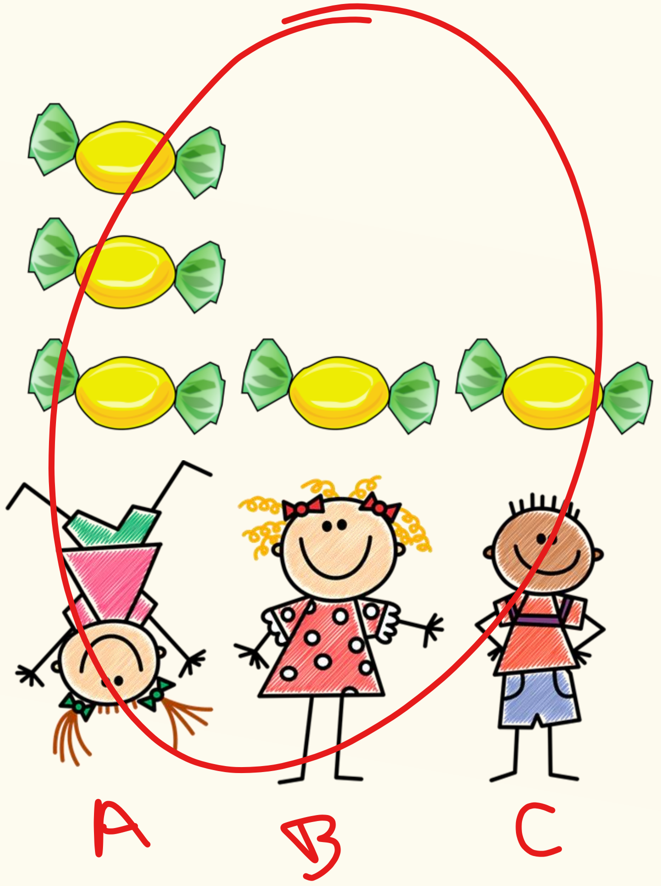


How many ways can we give five **indistinguishable** candies to these three kids?

Kids + Candies

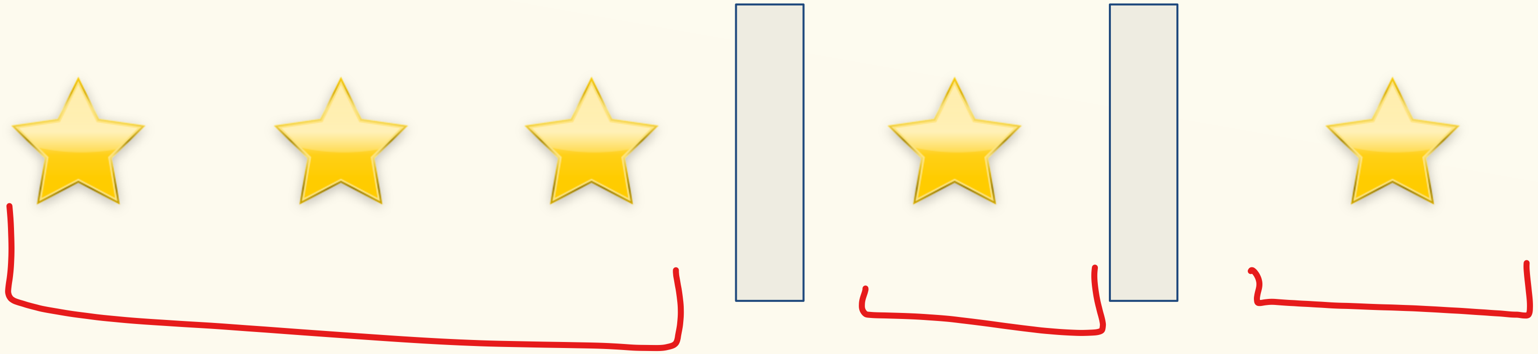


Kids + Candies



Idea: Count something equivalent

5 “stars” for candies, 2 “bars” for dividers.



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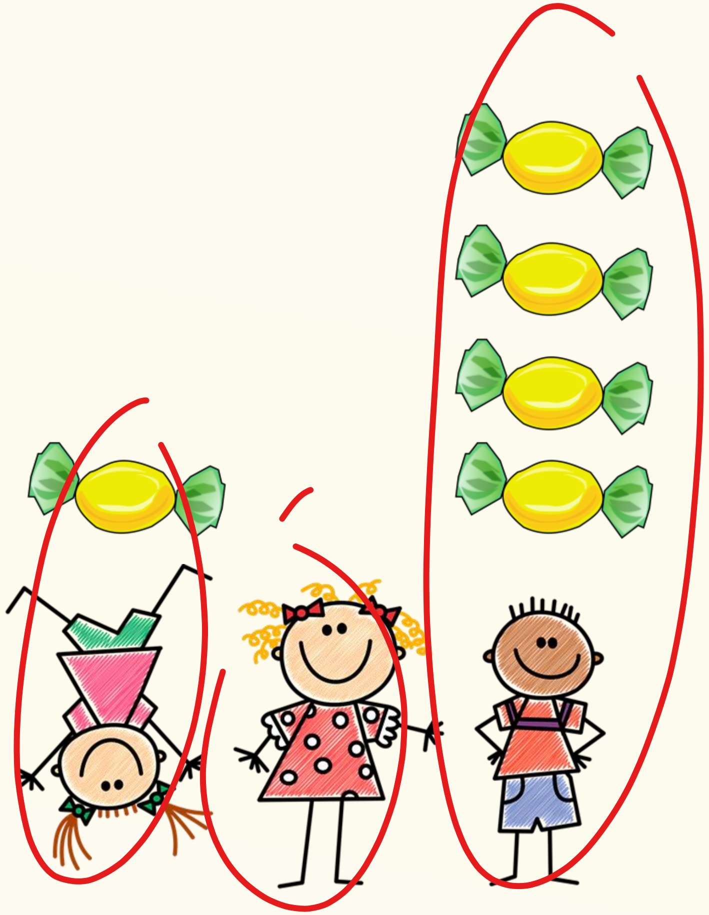
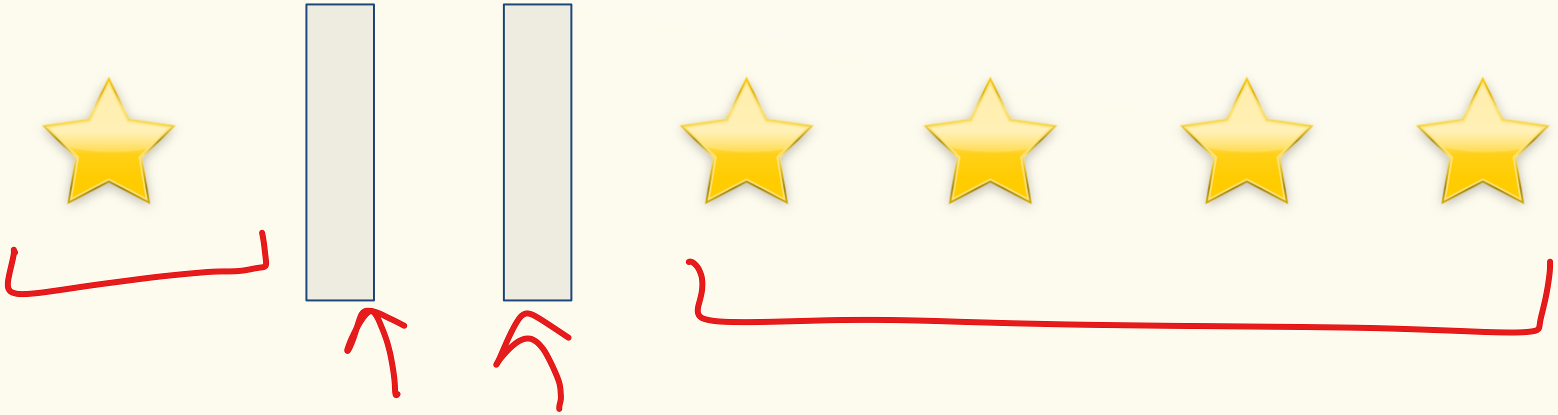
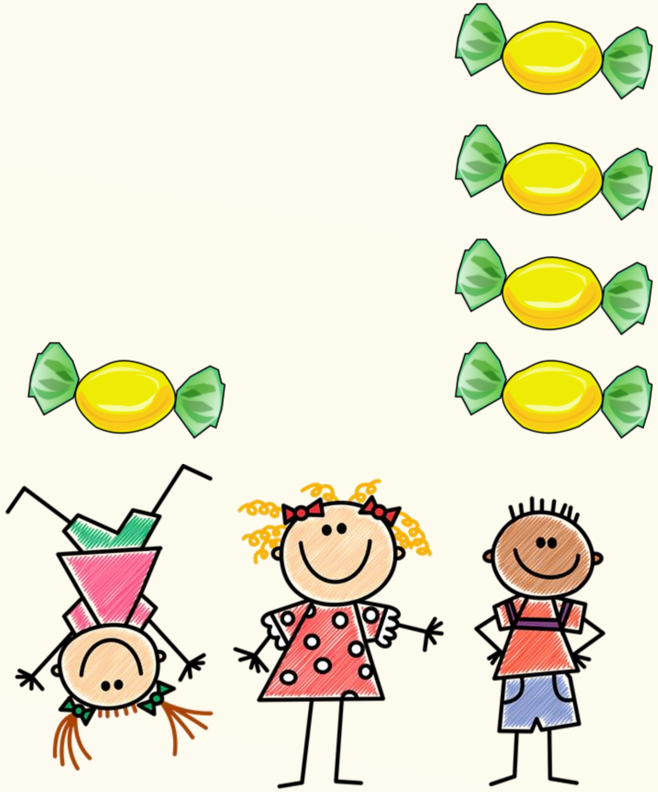




Kids + Candies



For each candy distribution, there is exactly one corresponding way to arrange the stars and bars.

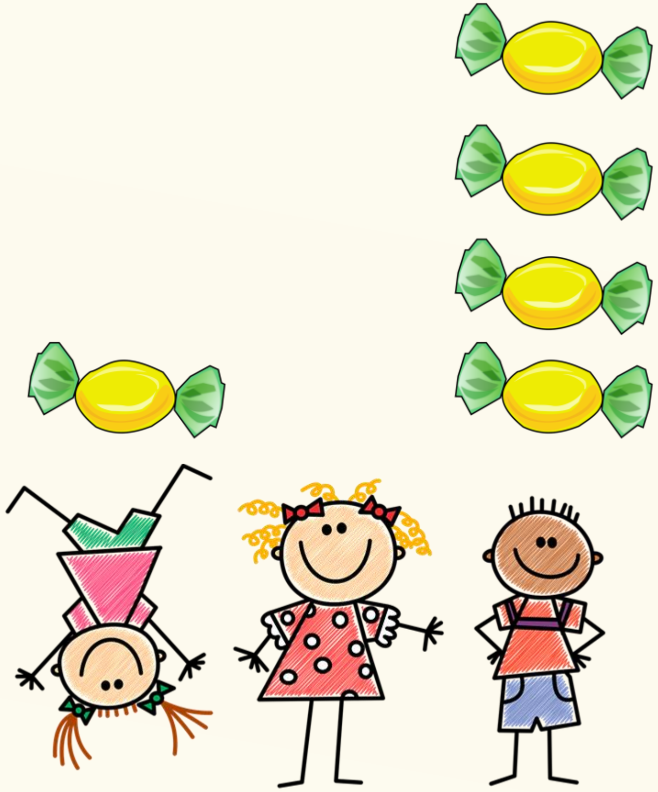
Conversely, for each arrangement of stars and bars, there is exactly one candy distribution it represents.

Kids + Candies

Hence, the number of ways to distribute candies to the 3 kids is the number of arrangements of stars and bars.

This is

$$\binom{7}{2} = \binom{7}{5}$$



Stars and Bars / Divider method

n = 5

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

k = 3

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

_ _ | _ | _ _

5 + 3 - 1

Example – Sum of integers

"How many solutions $(x_1, \ldots, x_k)$ *such that* $x_1, \ldots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

Example: $k = 3$, $n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots$

Example – Sum of integers

Example: $k = 3, n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots$

Clever representation of solutions

$(3,1,1)$	$(2,1,2)$	$(1,0,4)$
↓	↓	↓
1 1 1 0 1 0 1	1 1 0 1 0 1 1	1 0 0 1 1 1 1

Example – Sum of integers

Example: $k = 3, n = 5$

sols = # strings from $\{0,1\}^7$ w/ exactly two 0s $= \binom{7}{2} = 21$

Clever representation of solutions

$(3,1,1)$ ↓ $1\ 1\ 1\ 0\ 1\ 0\ 1$

$(2,1,2)$ ↓ $1\ 1\ 0\ 1\ 0\ 1\ 1$

$(1,0,4)$ ↓ $1\ 0\ 0\ 1\ 1\ 1\ 1$

Example – Sum of integers

"How many solutions $(x_1, \ldots, x_k)$ *such that* $x_1, \ldots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

sols = # strings from $\{0,1\}^{n+k-1}$ w/ $k-1$ 0s

$$= \binom{n+k-1}{k-1}$$

After a change in representation, the problem magically reduces to counting combinations.

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- **How to Answer a Question**

How to Answer a Question

- Be unambiguous
- Show that you understand the material
- Use notation correctly and aptly
- Don’t be unnecessarily verbose

- “A classmate, who hasn’t solved that problem but is up-to-date on the material, should be able to read your solution and be reasonably convinced that it is the correct answer.”

Example Solution (Section 1, Problem 10)

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the n*m people be arranged if members of a family must stay together?

Use the product rule. First, arrange the n families in an order. This is a permutation of n distinct elements, so there are n! ways to do this. Then, arrange the family members within each family, which is also a permutation so m!. There are n families so we get n! * (m!)^n