CSE 312: Foundations of Computing II

Section 3: Conditional Probability, Bayes Theorem

1. Review of Main Concepts

(a) **Conditional Probability** (only defined when $Pr(B) > 0$) $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

(b) **Independence**: Events E and F are independent iff $\mathbb{P}(E \cap F) = \mathbb{P}(E)\mathbb{P}(F)$, or equivalently $\mathbb{P}(F) = \mathbb{P}(F|E)$, or equivalently $\mathbb{P}(E) = \mathbb{P}(E|F)$

(c) **Bayes Theorem**: $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$

(d) **Partition:** Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff

- $E_1, \ldots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
- $E_1, \ldots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$

(e) **Law of Total Probability (LTP)**: Suppose $A_1, \ldots, A_n$ partition Ω and let B be any event. Then $\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \cap A_i) = \sum_{i=1}^{n} \mathbb{P}(\text{B} \mid A_i)\mathbb{P}(A_i)$

(f) **Bayes Theorem with LTP**: Suppose $A_1, \ldots, A_n$ partition Ω and let B be any event. Then $\mathbb{P}(A_1|B) = \dfrac{\mathbb{P}(B \mid A_1)\mathbb{P}(A_1)}{\sum_{i=1}^{n} \mathbb{P}(\text{B} \mid A_i)\mathbb{P}(A_i)}$. In particular, $\mathbb{P}(A|B) = \dfrac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(\text{B} \mid A)\mathbb{P}(A) + \mathbb{P}(\text{B} \mid A^C)\mathbb{P}(A^C)}$

2. Naive Bayes Presentation

Some parts of section03 may be a presentation on Naive Bayes... pending further updates!

3. Random Grades?

Suppose there are three possible teachers for CSE 312: Aleks Jovcic, Anna Karlin, and Shayan Oveis Gharan. Suppose the probabilities of getting an A in Aleks's class is $\frac{5}{15}$, for Anna's class is $\frac{3}{15}$, and for Shayan's class is $\frac{1}{15}$. Suppose you are assigned a grade randomly according to the given probabilities when you take a class from one of these professors, irrespective of your performance. Furthermore, suppose Shayan teaches your class with probability $\frac{1}{2}$ and Anna and Aleks have an equal chance of teaching if Shayan isn't. What is the probability you had Shayan, given that you received an A? Compare this to the unconditional probability that you had Shayan.

4. Coin Flipping

Suppose we have a coin with probability p of heads. Suppose we flip this coin n times independently. Let X be the number of heads that we observe. What is $\mathbb{P}(X = k)$, for $k = 0, \ldots n$? Verify that $\sum_{k=0}^{n} \mathbb{P}(X = k) = 1$, as it should.

5. More Coin Flipping

Suppose we have a coin with probability p of heads. Suppose we flip this coin independently until we flip a head for the first time. Let X be the number of times we flip the coin *up to and including* the first head. What is $\mathbb{P}(X = k)$, for $k = 1, 2, \ldots$? Verify that$\sum_{k=1}^{\infty} \mathbb{P}(X = k) = 1$, as it should. (You may use the fact that $\sum_{j=0}^{\infty} a^j = \frac{1}{1-a}$ for$|a| < 1$).

6. Game Show

Corrupted by their power, the judges running the popular game show America's Next Top Mathematician have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, she will be allowed to stay with probability 1. If the contestant has not been bribing the judges, she will be allowed to stay with probability 1/3, independent of what happens in earlier episodes. Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.

(a) If you pick a random contestant, what is the probability that she is allowed to stay during the first episode?

(b) If you pick a random contestant, what is the probability that she is allowed to stay during both episodes?

(c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she gets kicked off during the second episode?

(d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that she was bribing the judges?

7. No More Coins Please

There are three coins, C_1, C_2, and C_3. The probability of "heads" is 1 for C_1, 0 for C_2, and p for C_3. A coin is picked among these three uniformly at random, and then flipped a certain number of times.

(a) What is the probability that the first n flips are tails?

(b) Given that the first n flips were tails, what is the probability that C_1 was flipped / C_2 was flipped / C_3 was flipped?

8. Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

(a) What is the probability the system is functioning?

(b) If the system is functioning, what is the probability that component 1 is working?

(c) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

9. Marbles in Pockets

A girl has 5 blue and 3 white marbles in her left pocket, and 4 blue and 4 white marbles in her right pocket. If she transfers a randomly chosen marble from left pocket to right pocket without looking, and then draws a randomly chosen marble from her right pocket, what is the probability that it is blue?

10. Allergy Season

In a certain population, everyone is equally susceptible to colds. The number of colds suffered by each person during each winter season ranges from 0 to 4, with probability 0.2 for each value (see table below). A new cold prevention drug is introduced that, for people for whom the drug is effective, changes the probabilities as shown in the table. Unfortunately, the effects of the drug last only the duration of one winter season, and the drug is only effective in 20% of people, independently.

number of colds	no drug or ineffective	drug effective
0	0.2	0.4
1	0.2	0.3
2	0.2	0.2
3	0.2	0.1
4	0.2	0.0

(a) Sneezy decides to take the drug. Given that he gets 1 cold that winter, what is the probability that the drug is effective for Sneezy?

(b) The next year he takes the drug again. Given that he gets 2 colds in this winter, what is the updated probability that the drug is effective for Sneezy?

(c) Why is the answer to (b) the same as the answer to (a)?

11. Infinite Lottery

Suppose we randomly generate a number from the natural numbers $\mathbb{N} = \{1, 2, \ldots\}$. Let A_k be the event we generate the number k, and suppose $\mathbb{P}(A_k) = (\frac{1}{2})^k$. Once we generate a number k, that is the maximum we can win. That is, after generating a value k, we can win any number in $[k] = \{1, ..., k\}$ dollars. Suppose the probability that we win $\$j$ for $j \in [k]$ is "uniform", that is, each has probability $\frac{1}{k}$. Let B be the event we win exactly $\$1$. Given that we win exactly one dollar, what is the probability that the number generated was also 1? You may use the fact that $\sum_{j=1}^{\infty} \frac{1}{j \cdot a^j} = \ln(\frac{a}{a-1})$for $a > 1$.