CSE 312: Foundations of Computing II

Section 2: Intro Probability

1. Review of Main Concepts

(a) **Number of ways to select from n distinct objects:**

(a) **Permutations** (number of ways to linearly arrange k objects out of n distinct objects, when the order of the k objects matters):

$$P(n, k) = \frac{n!}{(n-k)!}$$

(b) **Combinations** (number of ways to choose k objects out of n distinct objects, when the order of the k objects does not matter):

$$\frac{n!}{k!(n-k)!} = \binom{n}{k} = C(n, k)$$

(b) **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.

(c) **Binomial Theorem:** $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$

(d) **Principle of Inclusion-Exclusion (PIE)**: For 2 events, it says $|A \cup B| = |A| + |B| - |A \cap B|$
For 3 events: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
In general: +singles - doubles + triples - quads + ...

(e) **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.

(f) **Multinomial coefficients**: Suppose there are n objects, but only k are distinct, with $k \leq n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is:

$$\frac{n!}{n_1! n_2! \cdots n_k!} = \binom{n}{n_1, n_2, \ldots, n_k}$$

(g) **Pigeonhole Principle**: Suppose there are $n-1$ pigeon holes and n pigeons, and each pigeon goes into a hole. Then, there must be some hole that has two pigeons in it. This simple observation is surprisingly useful in computer science.

We can put this more generally as: if there are n pigeons and k holes, and $n > k$, some hole has at least $\lceil \frac{n}{k} \rceil$ pigeons.

For the pigeon haters out there, we can also express this as "we have n holes and $n-1$ pigeons...". Pick your favorite.

(h) **Combinatorial proof:** Prove identity by showing that there are two different ways of counting some set of objects.

(i) **Key Probability Definitions**

(a) **Sample Space:** The set of all possible outcomes of an experiment, denoted Ω or S

(b) **Event:** Some subset of the sample space, usually a capital letter such as $E \subseteq \Omega$

(c) **Union:** The union of two events E and F is denoted $E \cup F$

(d) **Intersection:** The intersection of two events E and F is denoted $E \cap F$ or EF

(e) **Mutually Exclusive:** Events E and F are mutually exclusive iff $E \cap F = \emptyset$

(f) **Complement:** The complement of an event E is denoted E^C or $\overline{E}$ or $\neg E$, and is equal to $\Omega \setminus E$

(g) **DeMorgan's Laws:** $(E \cup F)^C = E^C \cap F^C$ and $(E \cap F)^C = E^C \cup F^C$

(h) **Probability of an event E:** denoted $\mathbb{P}(E)$ or $\Pr(E)$ or $P(E)$

(i) **Partition:** Nonempty events $E_1, \ldots, E_n$ partition the sample space Ω iff

- $E_1, \ldots, E_n$ are exhaustive: $E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$, and
- $E_1, \ldots, E_n$ are pairwise mutually exclusive: $\forall i \neq j, E_i \cap E_j = \emptyset$
  - Note that for any event A (with $A \neq \emptyset, A \neq \Omega$): A and A^C partition Ω

(j) **Axioms of Probability and their Consequences**

(a) **Axiom 1: Non-negativity** For any event E, $\mathbb{P}(E) \geq 0$

(b) **Axiom 2: Normalization** $\mathbb{P}(\Omega) = 1$

(c) **Axiom 3: Countable Additivity** If E and F are mutually exclusive, then $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F)$. Also, if $E_1, E_2, \ldots$ is a countable sequence of disjoint events, $\mathbb{P}(\bigcup_{k=1}^{\infty} E_i) = \sum_{k=1}^{\infty} \mathbb{P}(E_i)$.

(d) **Corollary 1: Complementation** $\mathbb{P}(E) + \mathbb{P}(E^C) = 1$

(e) **Corollary 2: Monotonicity** If $E \subseteq F$, $\mathbb{P}(E) \leq \mathbb{P}(F)$

(f) **Corollary 2: Inclusion-Exclusion** $\mathbb{P}(E \cup F) = \mathbb{P}(E) + \mathbb{P}(F) - \mathbb{P}(E \cap F)$

(k) **Equally Likely Outcomes:** If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) = \dfrac{|E|}{|\Omega|}$.

- Make sure to be consistent when counting $|E|$ and $|\Omega|$. Either order matters in both, or order doesn't matter in both.

2. HBCDEFGA

How many ways are there to permute the 8 letters A, B, C, D, E, F, G, H so that A is not at the beginning and H is not at the end?

3. Spades and Hearts

Given 3 different spades and 3 different hearts, shuffle them. Compute $\Pr(E)$, where E is the event that the suits of the shuffled cards are in alternating order.

4. Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says that each kid should take exactly n candies. Suppose that when the next kid shows up, they draw n candies, with each subset of size n equally likely to be drawn. What is the probability the kid draws exactly k kit kats?

5. Staff Photo

Suppose we have 9 chairs (in a row) with 6 TA's, and instructors Aleks, Anna, and Shayan to be seated. Suppose all seatings are equally likely. What is the probability that every instructor has a TA to their immediate left and right?

6. A Team and a Captain

Give a combinatorial proof of the following identity:

$$n\binom{n-1}{r-1} = \binom{n}{r}r.$$

Hint: Consider two ways to choose a team of size r out of a set of size n and a captain of the team (who is also one of the team members).

7. Binomials

What is the coefficient of z^{36} in $(-2x^2yz^3 + 5uv)^{312}$?

8. Weighted Die

Consider a weighted die such that

- $\text{Pr}(1) = \text{Pr}(2)$,
- $\text{Pr}(3) = \text{Pr}(4) = \text{Pr}(5) = \text{Pr}(6)$, and
- $\text{Pr}(1) = 3\text{Pr}(3)$.

What is the probability that the outcome is 3 or 4?

9. Fleas on Squares (Pigeonhole principle)

25 fleas sit on a 5×5 checkerboard, one per square. At the stroke of noon, all jump across an edge (not a corner) of their square to an adjacent square. At least two must end up in the same square. Why?

10. PigONEholes

Let $k \geq 2$ be some integer. Show that there exists a positive integer n consisting of only digits $0, 1$ and no larger than 10^{k+2} such that $k|n$. (Hint: Consider the sequence of length $k+1$ of $1, 11, 111, 1111, ...$).

11. Ingredients

(a) Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

(b) Repeat the question for the letters "AAAAABBB".

12. Divisibility

Consider the set $T = \{1, 2, ..., 36050\}$, and suppose we choose a subset S of size 3605, each equally likely. What is the probability that there are two (distinct) numbers in S whose difference is divisible by 99?

13. Congressional Tea Party

Twenty politicians are having a tea party, 6 Democrats and 14 Republicans.

(a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans?

(b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats?

14. Friendly Proofs

Show that in a group of n people (who may be friends with any number of other people), two must have the same number of friends.

15. Count the Solutions

Consider the following equation: $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 = 70$. A solution to this equation over the nonnegative integers is a choice of a nonnegative integer for each of the variables $a_1, a_2, a_3, a_4, a_5, a_6$ that satisfies the equation. For example, $a_1 = 15, a_2 = 3, a_3 = 15, a_4 = 0, a_5 = 7, a_6 = 30$ is a solution. To be different, two solutions have to differ on the value assigned to some a_i. How many different solutions are there to the equation?