CSE 312: Foundations of Computing II

Section 1: Counting I Solutions

1. Summary of Main Concepts (Counting I)

(a) **Sum Rule:** If an experiment can either end up having one of N outcomes, or one of M outcomes (where there is no overlap), then the total number of possible outcomes is $N + M$.

(b) **Product Rule:** If an object or an outcome can be selected by a sequence of choices, where there are m_1 possibilities for the first choice, m_2 possibilities for the second choice (given the first), m_3 possibilities for the third choice (given the first two) and so on up to the k-th choice, then there are $m_1 \cdot m_2 \cdot m_3 \cdots m_k = \prod_{i=1}^{k} m_i$ possible outcomes overall.

(c) **Number of ways to order n distinct objects:** $n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1$

(d) **Complementary Counting (Complementing):** If asked to find the number of ways to do X, you can: find the total number of ways and then subtract the number of ways to not do X.

2. Sets

a) For each one of the following sets, give its cardinality, i.e., indicate how many elements it contains:

- $A = \emptyset$
- $B = \{\emptyset\}$
- $C = \{\{\emptyset\}\}$
- $D = \{\emptyset, \{\emptyset\}\}$

Solution:

- $A = 0$
- $B = 1$
- $C = 1$
- $D = 2$

b) Let $S = \{a, b, c\}$ and $T = \{c, d\}$. Compute:

- $S \cup T$
- $S \cap T$
- $S \setminus T$
- $2^{S \setminus T}$
- $S \times T$

Solution:

- $S \cup T = \{a, b, c, d\}$
- $S \cap T = \{c\}$
- $S \setminus T = \{a, b\}$
- $2^{S \setminus T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- $S \times T = \{(a, c), (a, d), (b, c), (b, d), (c, c), (c, d)\}$

3. Basic Counting

a) Credit-card numbers are made of 15 decimal digits, and a 16th checksum digit (which is *uniquely determined* by the first 15 digits). How many credit-card numbers are there?

Solution:

10^{15}

b) How many positive divisors does $1440 = 2^5 3^2 5$ have?

Solution:

$6 \cdot 3 \cdot 2 = 36$.

Every positive divisor of 1440 can be written as $2^i 3^j 5^k$ where $i \in \{0, \ldots, 5\}$, $j \in \{0, 1, 2\}$, and $k \in \{0, 1\}$.

c) How many ways are there to arrange the CSE 312 staff on a line (Aleks and 6 TAs) for a group picture?

Solution:

$7!$.

d) There are 50 seats and 50 people (44 students, 6 TAs) in the classroom. Suppose that the back row contains exactly 6 seats and the 6 TAs must sit in the back row. How many seating arrangements are possible with this restriction?

Solution:

Use the product rule: First choose the 6 TAs' placement in the back row seats ($6!$ ways). Then arrange the remaining 44 students in the remaining 44 seats ($44!$ ways).

So the answer is $\boxed{6! \cdot 44!}$.

4. Seating

How many ways are there to seat 10 people, consisting of 5 couples, in a row of 10 seats if . . .

(a) . . . all couples are to get adjacent seats?

Solution:

Consider each couple as a unit. Apply the product rule, first choosing one of the $5!$ permutations of the 5 couples, and then, for each couple in turn, choosing one of the 2 permutations for how they sit (for a total of 2^5). Therefore, the answer is: $\boxed{5! \cdot 2^5.}$

(b) . . . anyone can sit anywhere, except that one couple insists on *not* sitting in adjacent seats?

Solution:

Apply complementary counting to first compute the total number of arrangements of the 10 people, and then subtract from this the number of arrangements in which that particular couple does get adjacent seats. There are $10!$ for the former, and there are $9! \cdot 2$ arrangements in which this couple does sit in adjacent seats, since you can treat the couple as a unit, permute the 9 "individuals" (consisting of 8 people plus the couple) and then consider the 2 permutations for that couple. That means the answer to the question is $\boxed{10! - 9! \cdot 2 = 8 \cdot 9!}$.

Alternatively, we can do casework. Name the two people in the couple A and B. There are two cases: A can sit on one of the ends, or not. If A sits on an end seat, A has 2 choices and B has 8 possible seats. If A doesn't sit on the end, A has 8 choices and B only has 7. So there are a total of $2 \cdot 8 + 8 \cdot 7$ ways A and B can sit. Once they do, the other 8 people can sit in $8!$ ways since there are no other restrictions. Hence the total number of ways is $\boxed{(2 \cdot 8 + 8 \cdot 7)8! = 9 \cdot 8 \cdot 8! = 8 \cdot 9!}$.

5. Weird Card Game

In how many ways can a pack of fifty-two cards be dealt to thirteen players, four to each, so that every player has one card of each suit?

Solution:

Apply the product rule: First deal the hearts, one to each person, then the spades, one to each person, then diamonds, then the clubs. For each of these steps, there are $13!$ possibilities. Therefore, the answer is $\boxed{13!^4}$.

6. Birthday Cake

A chef is preparing desserts for the week, starting on a Sunday. On each day, only one of five desserts (apple pie, cherry pie, strawberry pie, pineapple pie, and cake) may be served. On Thursday there is a birthday, so cake must be served that day. On no two consecutive days can the chef serve the same dessert. How many dessert menus are there for the week?

Solution:

Apply the product rule. Start from Thursday and work forward and backward in the week:
More precisely, given the 1 choice on Thursday, for each of Wednesday and Friday, there are 4 choices (the different pie options). Given the choice on Wednesday, there are 4 choices for Tuesday, and given the choice on Tuesday, there are 4 choices for Monday, and given the choice on Monday, there are 4 choices on Sunday. Similarly, given the choice on Friday, there are 4 choices on Saturday.
Therefore the answer is $4 \cdot 4 \cdot 4 \cdot 4 \cdot 1 \cdot 4 \cdot 4 = \boxed{4^6}$

7. Photographs

Suppose that 8 people, including you and a friend, line up for a picture. In how many ways can the photographer organize the line if she wants to have fewer than 2 people between you and your friend?

Solution:

This is most easily solved using the sum rule. Count the number of ways the line can be organized if you are next to your friend. Then count the number of ways the line can be organized if there is one person between you and your friend. Then use the sum rule to add these up.
Case 1: You are next to your friend. So we can think of you and your friend as being a "unit". Now apply the product rule: there are $7!$ ways to arrange the other 6 people together with the unit (of you and your friend). Once arranged, there are 2 ways to rearrange you and your friend in the order. So there are $7! \cdot 2$ ways to to line people up if you are next to your friend.
Case 2: There is exactly 1 person between you and your friend. Apply the product rule by first picking the person who is between you (6 choices). Then, thinking of you, your friend and that person as a "unit", consider all arrangements of the 5 people plus the unit ($6!$ ways). Finally, there are two ways for you and your friend to be placed within the trio. Therefore, altogether there are $6 \cdot 6! \cdot 2$ possibilities.
Therefore, the final answer is $\boxed{(2 \cdot 7 + 2 \cdot 6) \cdot 6!}$

8. Rabbits!

Rabbits Peter and Pauline have three offspring: Flopsie, Mopsie, and Cotton-tail. These five rabbits are to be distributed to four different pet stores so that no store gets both a parent and a child. It is not required that every store gets a rabbit. In how many different ways can this be done?

Solution:

Find the number of possibilities when Peter and Pauline go to the same store, and find the number of possibilities when they go to different stores, and then use the sum rule to get the final answer.
If Peter and Pauline go to the same store, there are 4 stores it could be. For each such choice, there are 3 choices of store for each of the 3 offspring, so 3^3 choices for all the offspring.
If Peter and Pauline go to different stores, there are $4 \cdot 3 = 12$ pairs of stores they could go to. For each such

choice, there are 2 choices of store for each of the 3 offspring, so 2^3 choices for all the offspring.

Therefore the answer is $\boxed{4 \cdot 3^3 + 12 \cdot 2^3}$.

9. Chickens and More Rabbits!

Our class has three pet rabbits, Flopsy, Mopsy, and Cottontail, as well as two pet chickens, Ginger and Rocky, but only 4 shelters to put them in. In how many different ways can we put these 5 pets in these 4 shelters so that no shelter has both a chicken and a rabbit?

Solution:

Find the number of possibilities when Ginger and Rocky go in the same shelter, and find the number of possibilities when they go to different terrariums, and then use the sum rule to get the final answer.

If Ginger and Rocky go in the same shelter, there are 4 shelter it could be. For each such choice, there are 3 choices of shelter for each of the 3 rabbits, so 3^3 choices for all the rabbits.

If Ginger and Rocky go in different shelteres, there are $4 \cdot 3 = 12$ pairs of shelters they could go in. For each such choice, there are 2 choices of shelters for each of the 3 rabbits, so 2^3 choices for all the rabbits.

Therefore the answer is $\boxed{4 \cdot 3^3 + 12 \cdot 2^3}$.

10. Extended Family Portrait

A group of n families, each with m members, are to be lined up for a photograph. In how many ways can the nm people be arranged if members of a family must stay together?

Solution:

Apply the product rule. First order the families; there are are $n!$ ways to do this. Then consider the families one by one and reorder their members. Within each family, there are $m!$ ways to order their members. So there are a total of $\boxed{n!(m!)^n}$ ways to line these people up according to the given constraints.