

**CSE 312**

# **Foundations of Computing II**

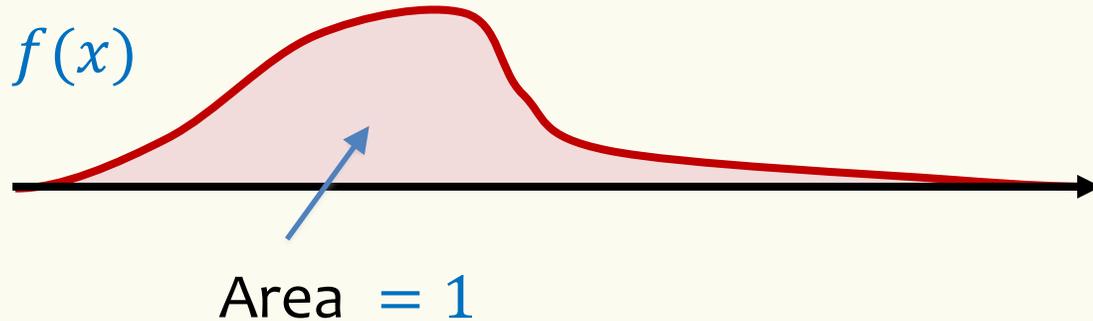
**Lecture 16: Normal Distribution & Central Limit Theorem**

# Review Continuous RVs

## Probability Density Function (PDF).

$f: \mathbb{R} \rightarrow \mathbb{R}$  s.t.

- $f(x) \geq 0$  for all  $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x) dx = 1$

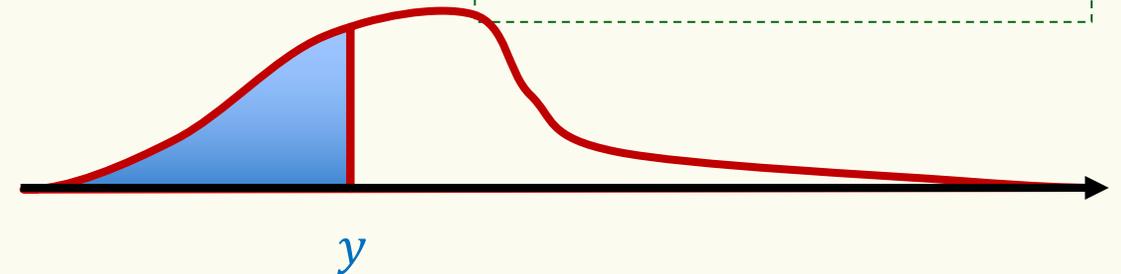


Density  $\neq$  Probability !

## Cumulative Distribution Function (CDF).

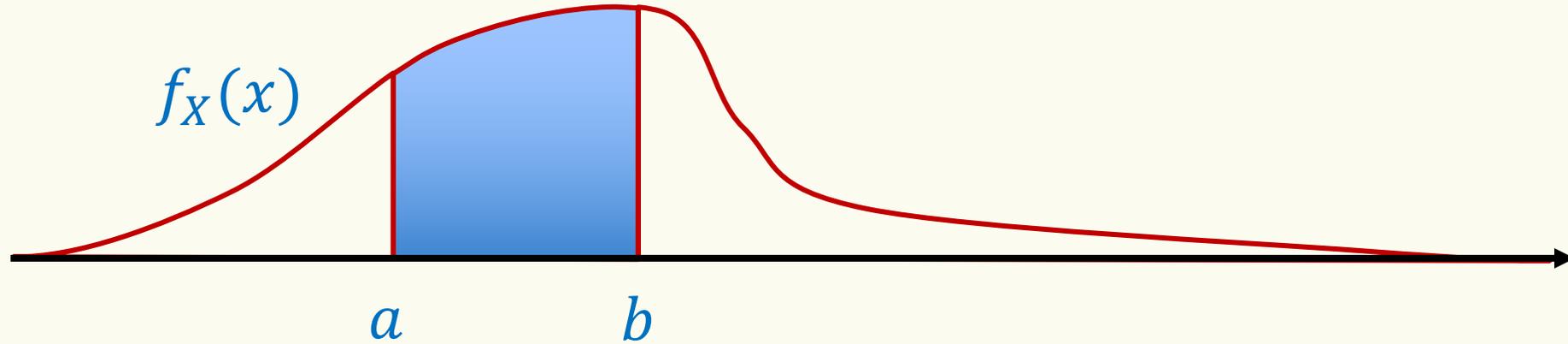
$$F(y) = \int_{-\infty}^y f(x) dx$$

Theorem.  $f(x) = \frac{dF(x)}{dx}$



$$F_X(y) = P(X \leq y)$$

## Review Continuous RVs



$$P(X \in [a, b]) = \int_a^b f_X(x) dx = F_X(b) - F_X(a)$$

# Review Exponential Distribution

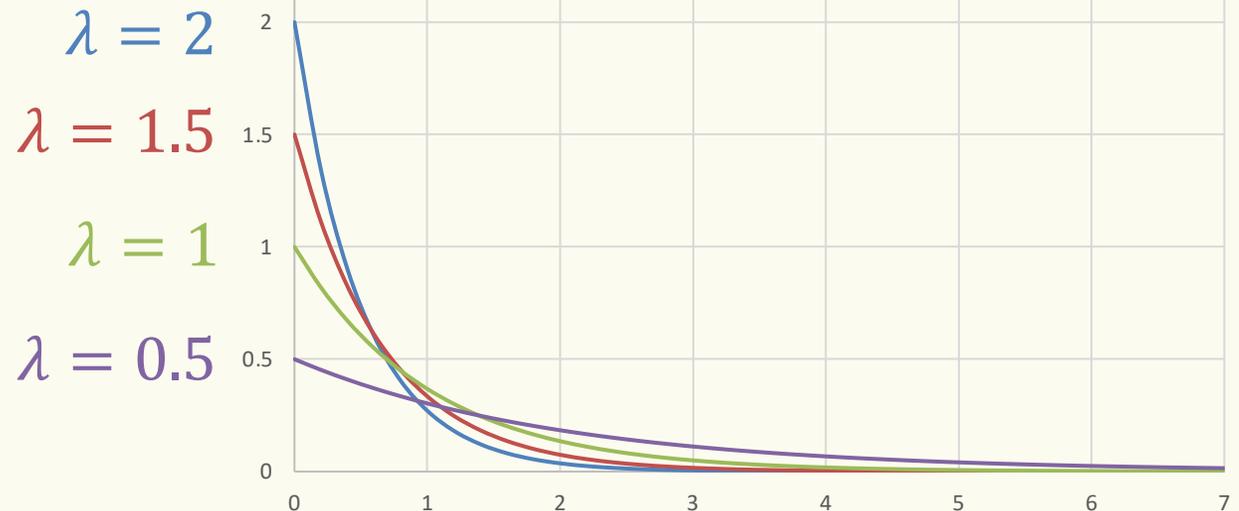
**Definition.** An **exponential random variable**  $X$  with parameter  $\lambda \geq 0$  is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write  $X \sim \text{Exp}(\lambda)$  and say  $X$  that follows the exponential distribution.

CDF: For  $y \geq 0$ ,

$$F_X(y) = 1 - e^{-\lambda y}$$



# Agenda

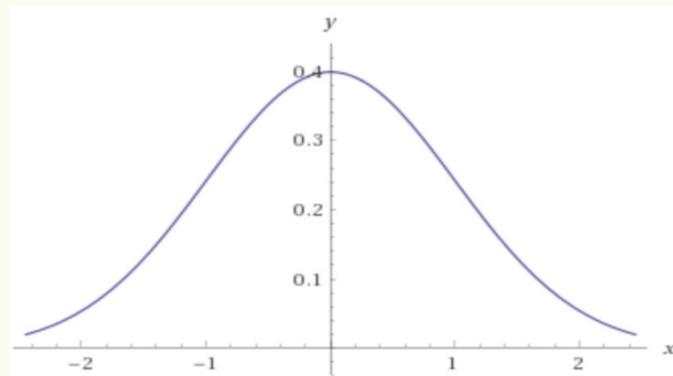
- Normal Distribution ◀
- Practice with Normals
- Central Limit Theorem (CLT)

# The Normal Distribution

**Definition.** A **Gaussian (or normal) random variable** with parameters  $\mu \in \mathbb{R}$  and  $\sigma \geq 0$  has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that  $X$  follows the Normal Distribution, and write  $X \sim \mathcal{N}(\mu, \sigma^2)$ .



$\mathcal{N}(0, 1)$ .



Carl Friedrich  
Gauss

# The Normal Distribution



Carl Friedrich  
Gauss

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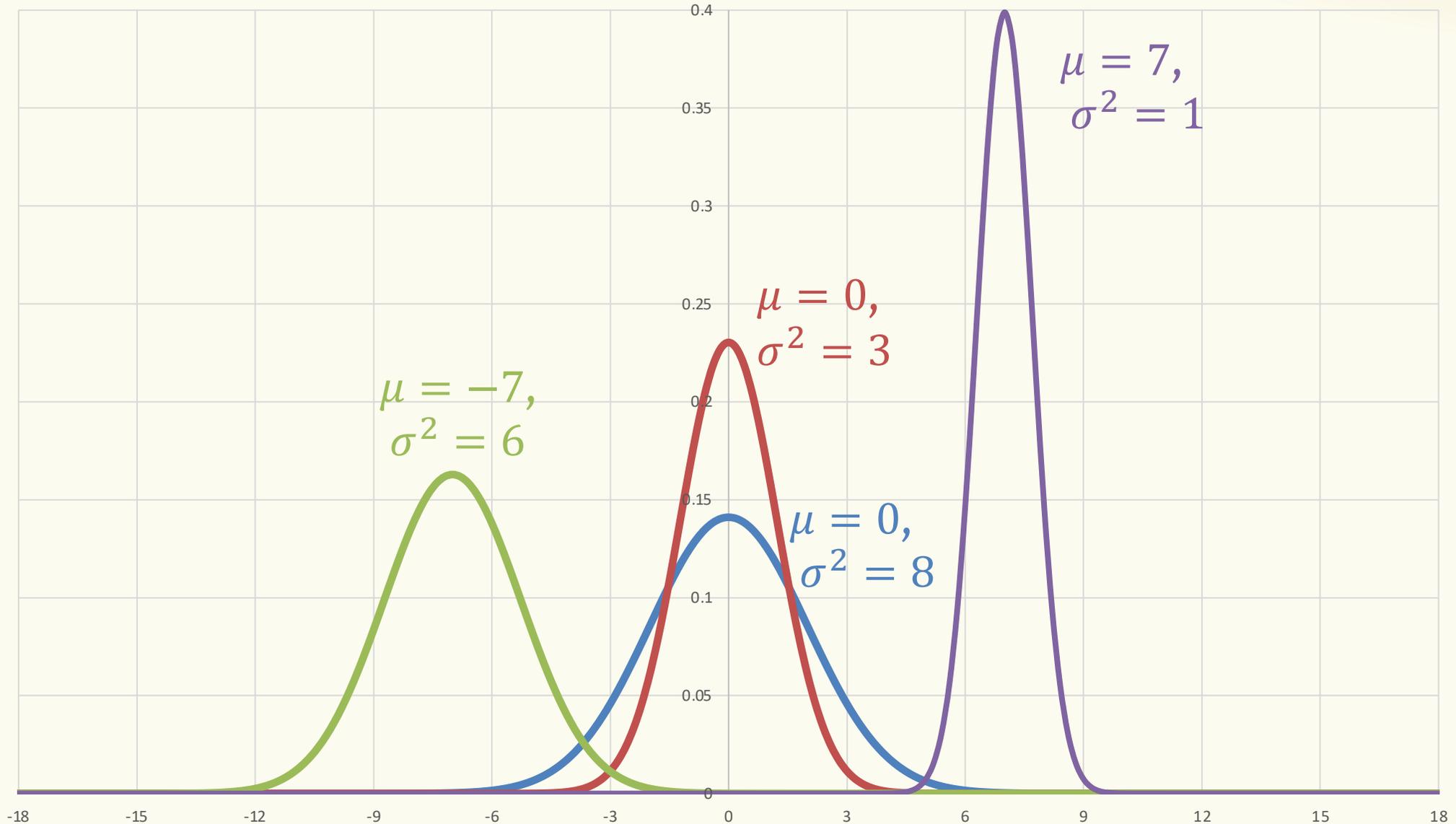
**Fact.** If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $\mathbb{E}[X] = \mu$ , and  $\text{Var}(X) = \sigma^2$

Proof of expectation is easy because density curve is symmetric around  $\mu$ ,

$f_X(\mu - x) = f_X(\mu + x)$ , but proof for variance requires integration of  $e^{-x^2/2}$

# The Normal Distribution

Aka a “Bell Curve” (imprecise name)



# Closure of normal distribution – Under Shifting and Scaling

**Fact.** If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

**Proof.**  $\mathbb{E}[Y] = a \mathbb{E}[X] + b = a\mu + b$

$$\text{Var}(Y) = a^2 \text{Var}(X) = a^2\sigma^2$$

Can show with algebra that the PDF of  $Y = aX + b$  is still normal.

Note:  $\frac{X-\mu}{\sigma} \sim \mathcal{N}(0, 1)$

# CDF of normal distribution

**Fact.** If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $Y = aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$

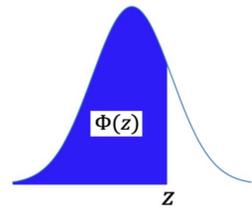
**Standard (unit) normal** =  $\mathcal{N}(0, 1)$

**CDF.**  $\Phi(z) = P(Z \leq z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^z e^{-x^2/2} dx$  for  $Z \sim \mathcal{N}(0, 1)$

Note:  $\Phi(z)$  has no closed form – generally given via tables

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $F_X(z) = P(X \leq z) = P\left(\frac{X-\mu}{\sigma} \leq \frac{z-\mu}{\sigma}\right) = \Phi\left(\frac{z-\mu}{\sigma}\right)$

# Table of Standard Cumulative Normal Density



$$P(Z \leq 1.09) = \Phi(1.09) \approx 0.8621$$

What is  
 $P(Z \leq -1.09)$ ?

**Poll:**

[pollev.com/stefanotessararo617](https://pollev.com/stefanotessararo617)

- a. 0.1379
- b. 0.8621
- c. 0
- d. Not able to compute

Φ Table:  $\mathbb{P}(Z \leq z)$  when  $Z \sim \mathcal{N}(0, 1)$

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.5279	0.53188	0.53586
0.1	0.53983	0.5438	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.6293	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.6591	0.66276	0.6664	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.7054	0.70884	0.71226	0.71566	0.71904	0.7224
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.7549
0.7	0.75804	0.76115	0.76424	0.7673	0.77035	0.77337	0.77637	0.77935	0.7823	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.8665	0.86864	0.87076	0.87286	0.87493	0.87698	0.879	0.881	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.9032	0.9049	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.9222	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.9452	0.9463	0.94738	0.94845	0.9495	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.9608	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.9732	0.97381	0.97441	0.975	0.97558	0.97615	0.9767
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.9803	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.983	0.98341	0.98382	0.98422	0.98461	0.985	0.98537	0.98574
2.2	0.9861	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.9884	0.9887	0.98899
2.3	0.98928	0.98956	0.98983	0.9901	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.9918	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.9943	0.99446	0.99461	0.99477	0.99492	0.99506	0.9952
2.6	0.99534	0.99547	0.9956	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.9972	0.99728	0.99736
2.8	0.99744	0.99752	0.9976	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.999

## Closure of the normal -- under addition

**Fact.** If  $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ ,  $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$  (both independent normal RV) then  $aX + bY + c \sim \mathcal{N}(a\mu_X + b\mu_Y + c, a^2\sigma_X^2 + b^2\sigma_Y^2)$

Note: The special thing is that **the sum of normal RVs is still a normal RV**.  
The values of the expectation and variance are **not** surprising.

### Why not?

- Linearity of expectation (always true)
- When  $X$  and  $Y$  are independent,  $\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y)$

# Agenda

- Normal Distribution
- Practice with Normals ◀
- Central Limit Theorem (CLT)

## What about Non-standard normal?

If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1)$

Therefore,

$$F_X(z) = P(X \leq z) = P\left(\frac{X - \mu}{\sigma} \leq \frac{z - \mu}{\sigma}\right) = \Phi\left(\frac{z - \mu}{\sigma}\right)$$

# Example

Let  $X \sim \mathcal{N}(0.4, 4 = 2^2)$ .

$$\begin{aligned} P(X \leq 1.2) &= P\left(\frac{X - 0.4}{2} \leq \frac{1.2 - 0.4}{2}\right) \\ &= P\left(\frac{X - 0.4}{2} \leq 0.4\right) = \Phi(0.4) \approx 0.6554 \end{aligned}$$

$\sim \mathcal{N}(0, 1)$

0.1	0.5398	0.5438
0.2	0.5793	0.5832
0.3	0.6179	0.6217
0.4	0.6554	0.6591
0.5	0.6915	0.6950
0.6	0.7257	0.7291
0.7	0.7580	0.7611

## Example

Let  $X \sim \mathcal{N}(3, 16)$ .

$$\begin{aligned} P(2 < X < 5) &= P\left(\frac{2 - 3}{4} < \frac{X - 3}{4} < \frac{5 - 3}{4}\right) \\ &= P\left(-\frac{1}{4} < Z < \frac{1}{2}\right) \\ &= \Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{4}\right) \\ &= \Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{4}\right)\right) \approx 0.29017 \end{aligned}$$

## Example – How Many Standard Deviations Away?

Let  $X \sim \mathcal{N}(\mu, \sigma^2)$ .

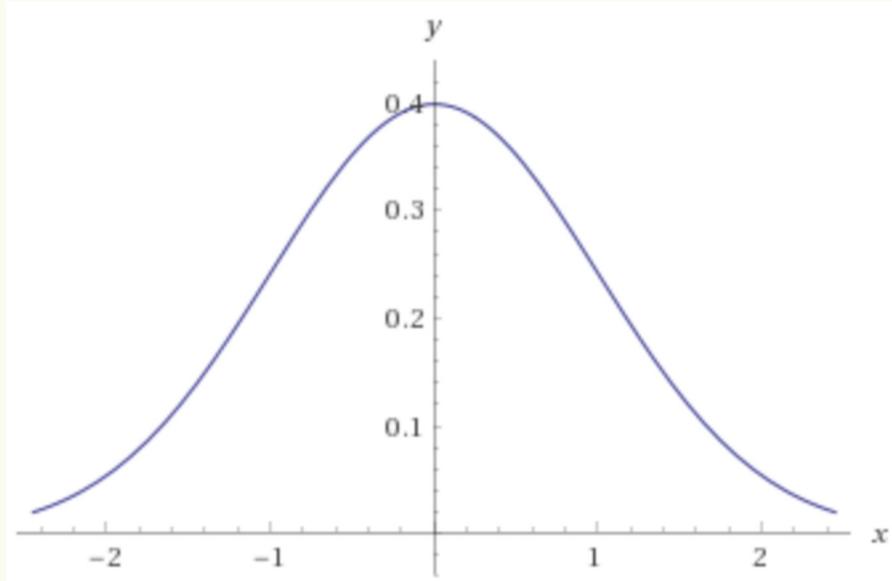
$$\begin{aligned} P(|X - \mu| < k\sigma) &= P\left(\frac{|X - \mu|}{\sigma} < k\right) = \\ &= P\left(-k < \frac{X - \mu}{\sigma} < k\right) = \Phi(k) - \Phi(-k) \end{aligned}$$

e.g.  $k = 1$ : 68%

$k = 2$ : 95%

$k = 3$ : 99%

# Brain Break



**Normal Distribution**



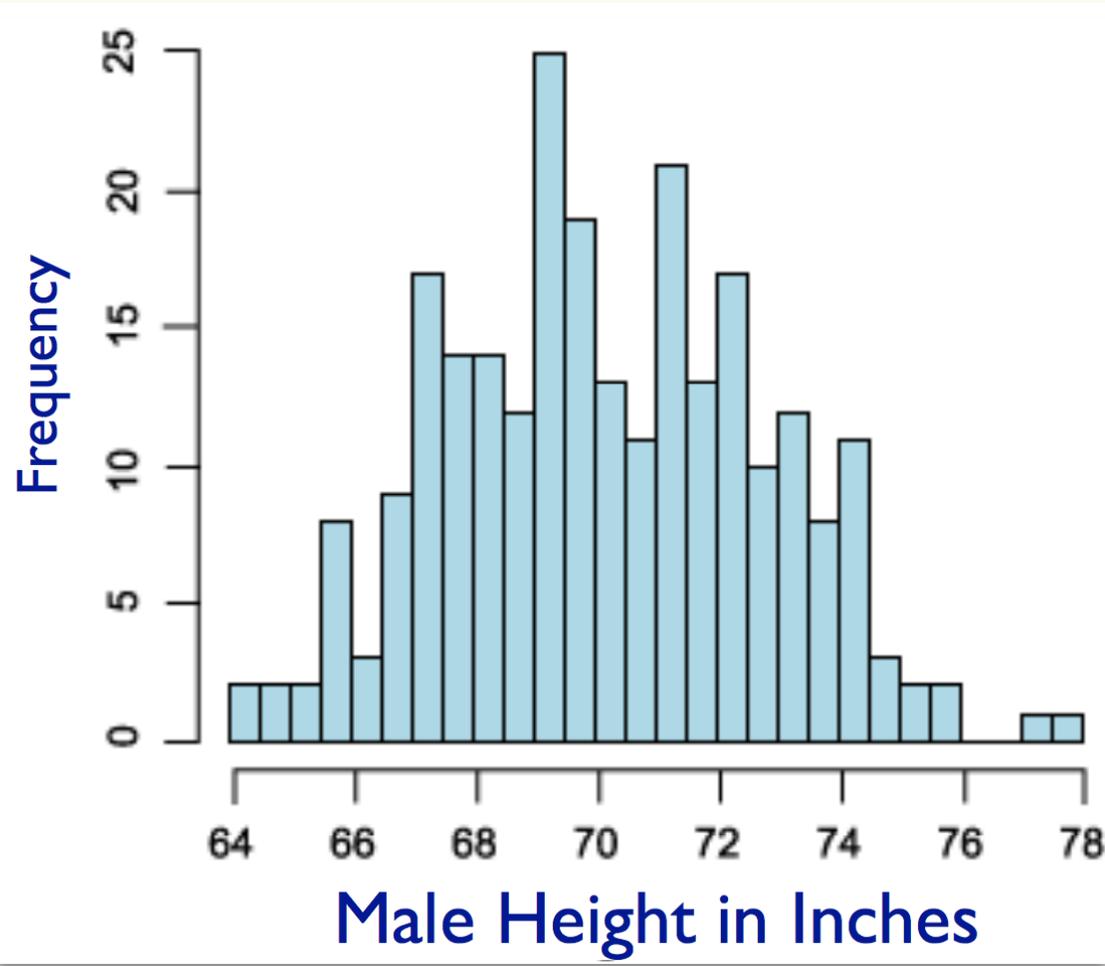
**Paranormal Distribution**

# Agenda

- Normal Distribution
- Practice with Normals
- Central Limit Theorem (CLT) ◀

# Gaussian in Nature

Empirical distribution of collected data often resembles a Gaussian ...



e.g. Height distribution resembles Gaussian.

R.A.Fisher (1918) observed that the height is likely the outcome of the sum of many independent random parameters, i.e., can be written as

$$X = X_1 + \dots + X_n$$

# Sum of Independent RVs

i.i.d. = independent and identically distributed

$X_1, \dots, X_n$  i.i.d. with expectation  $\mu$  and variance  $\sigma^2$

Define

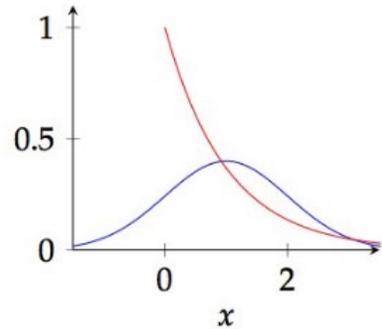
$$S_n = X_1 + \dots + X_n$$

$$\mathbb{E}[S_n] = \mathbb{E}[X_1] + \dots + \mathbb{E}[X_n] = n\mu$$

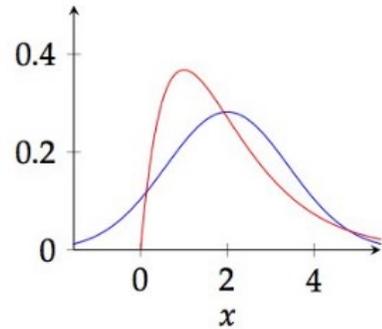
$$\text{Var}(S_n) = \text{Var}(X_1) + \dots + \text{Var}(X_n) = n\sigma^2$$

**Empirical observation:**  $S_n$  looks like a normal RV as  $n$  grows.

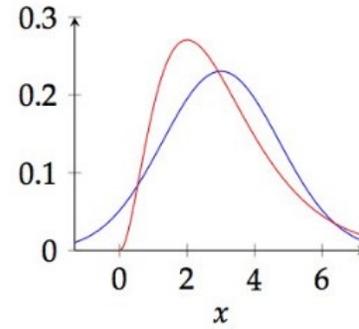
# Example: Sum of $n$ i.i.d. $\text{Exp}(1)$ random variables



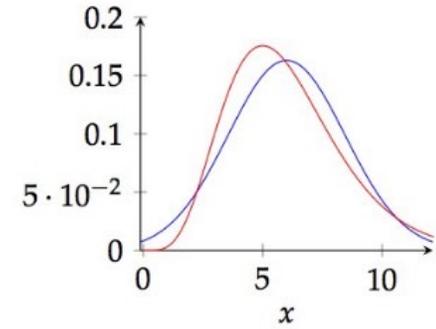
(a)  $n = 1$



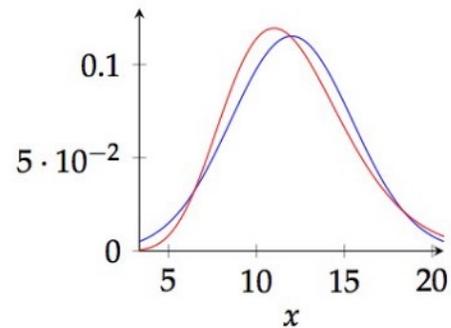
(b)  $n = 2$



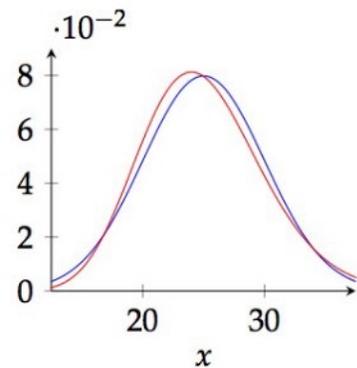
(c)  $n = 3$



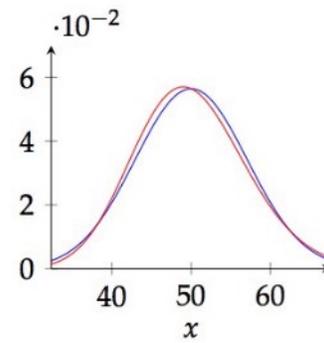
(d)  $n = 6$



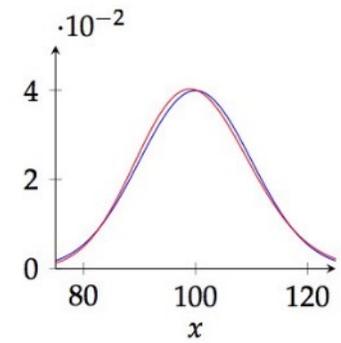
(e)  $n = 12$



(f)  $n = 25$

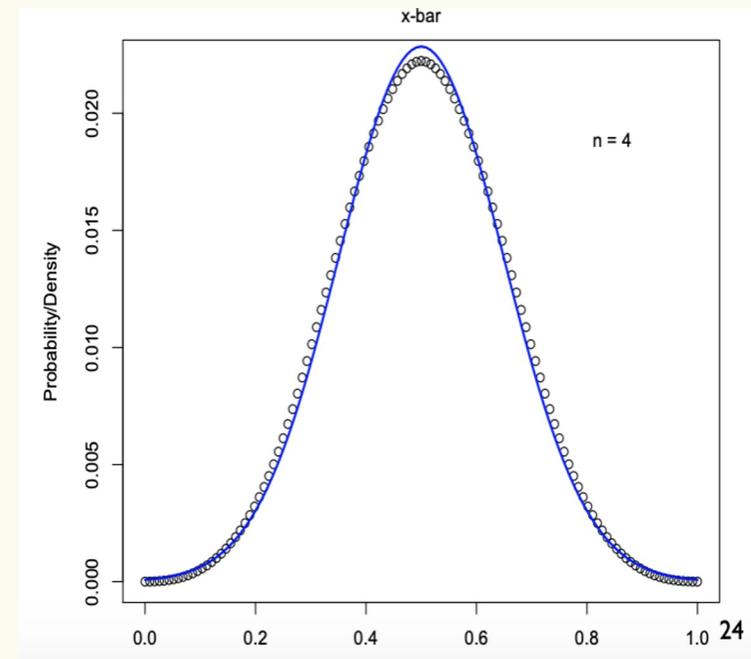
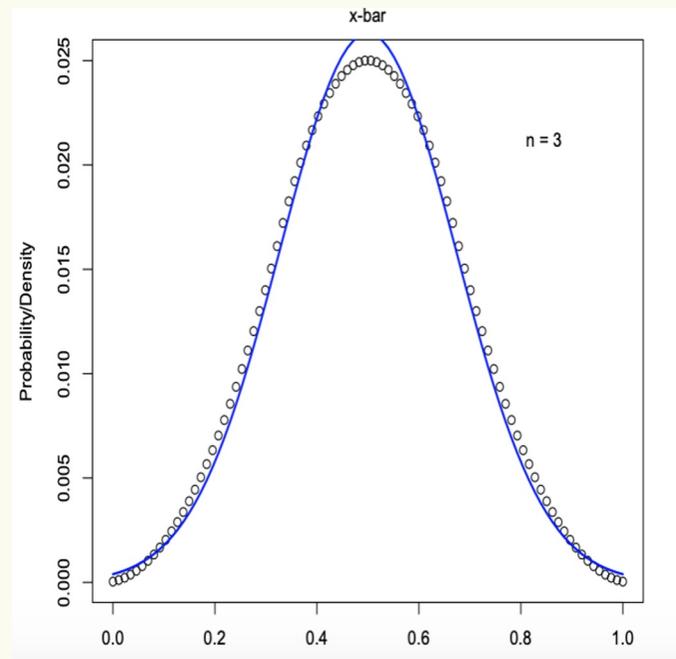
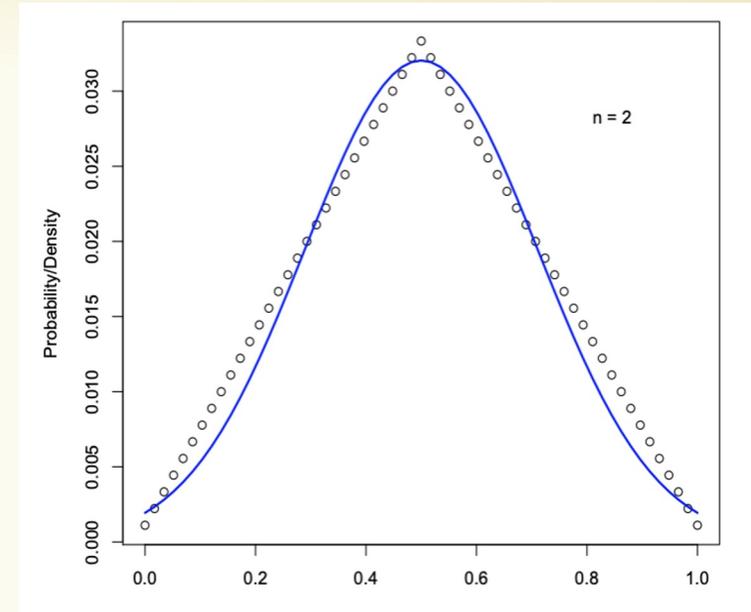
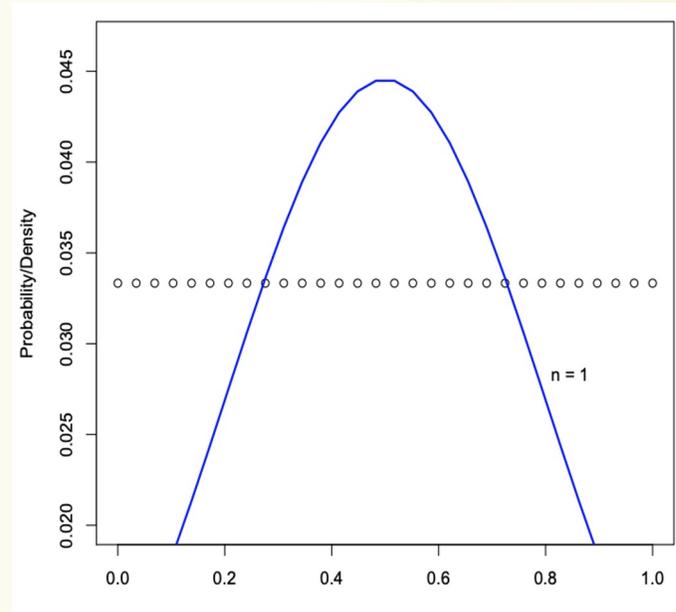


(g)  $n = 50$

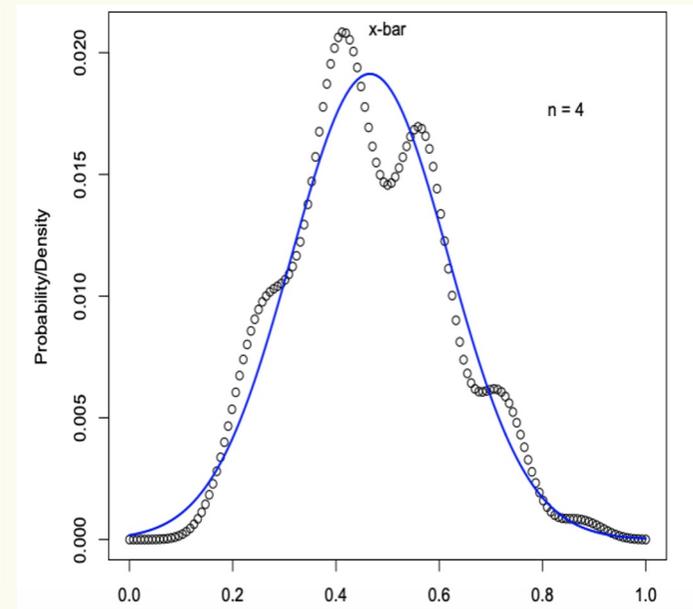
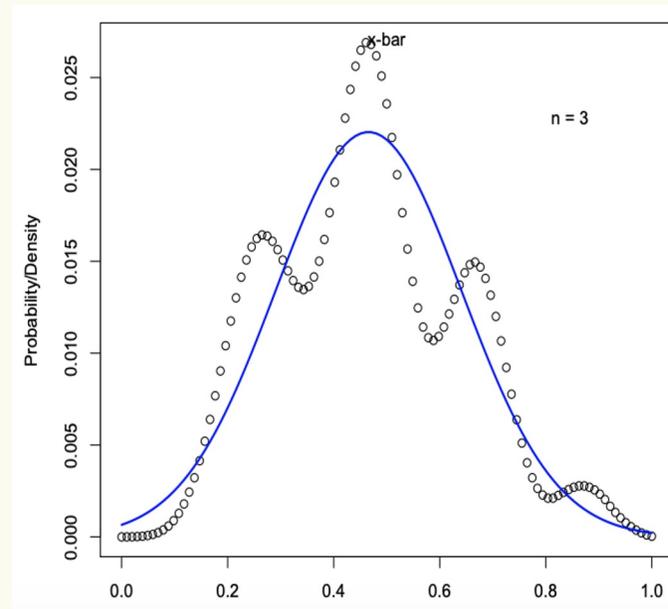
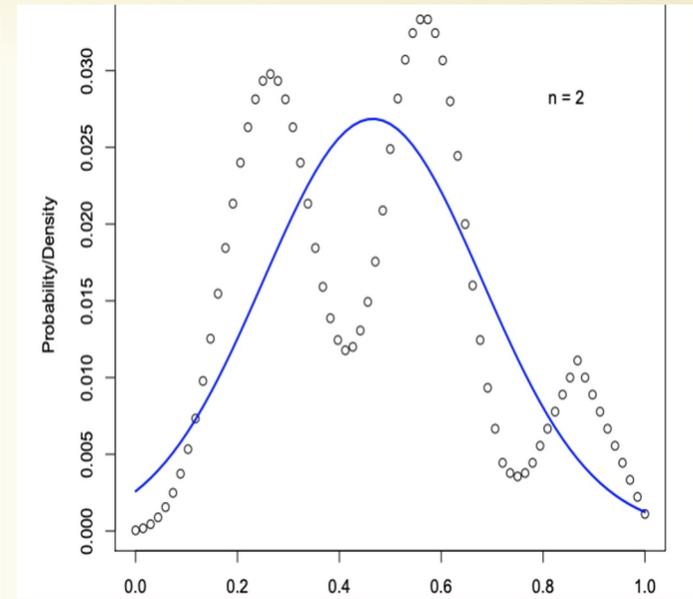
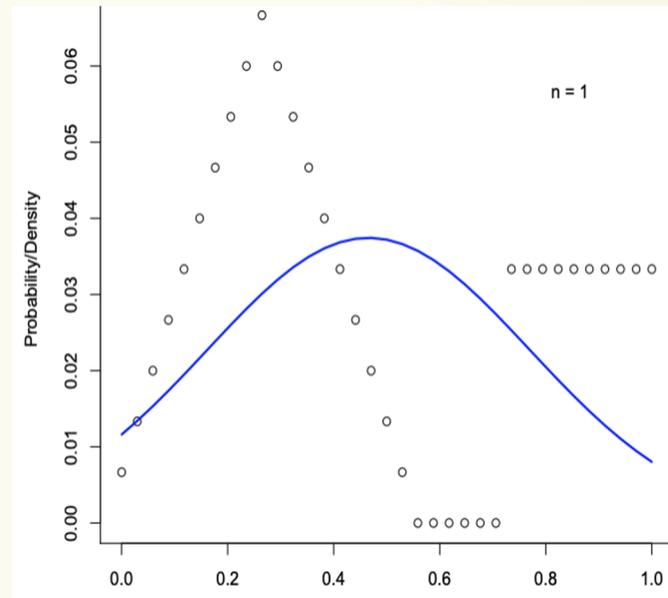


(h)  $n = 100$

# CLT (Idea)



# CLT (Idea)



# Central Limit Theorem

$X_1, \dots, X_n$  i.i.d., each with expectation  $\mu$  and variance  $\sigma^2$

Define  $S_n = X_1 + \dots + X_n$  and

$$Y_n = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

$$\mathbb{E}[Y_n] = \frac{1}{\sigma\sqrt{n}} (\mathbb{E}[S_n] - n\mu) = \frac{1}{\sigma\sqrt{n}} (n\mu - n\mu) = 0$$

$$\text{Var}(Y_n) = \frac{1}{\sigma^2 n} (\text{Var}(S_n - n\mu)) = \frac{\text{Var}(S_n)}{\sigma^2 n} = \frac{\sigma^2 n}{\sigma^2 n} = 1$$

# Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

**Theorem. (Central Limit Theorem)** The CDF of  $Y_n$  converges to the CDF of the standard normal  $\mathcal{N}(0,1)$ , i.e.,

$$\lim_{n \rightarrow \infty} P(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx$$

# Central Limit Theorem

$$Y_n = \frac{X_1 + \cdots + X_n - n\mu}{\sigma\sqrt{n}}$$

**Theorem. (Central Limit Theorem)** The CDF of  $Y_n$  converges to the CDF of the standard normal  $\mathcal{N}(0,1)$ , i.e.,

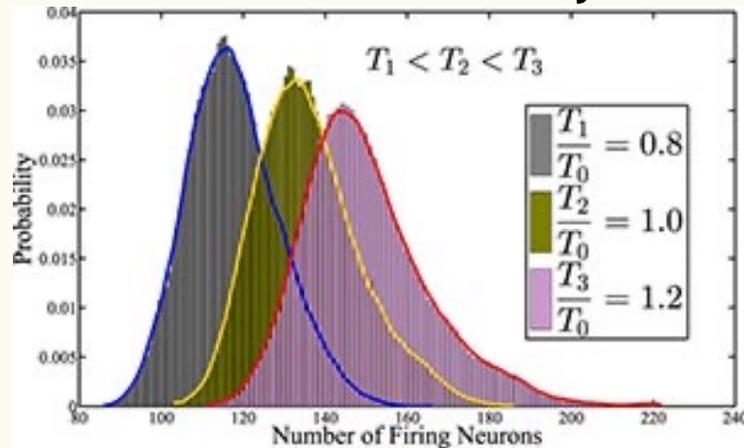
$$\lim_{n \rightarrow \infty} P(Y_n \leq y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-x^2/2} dx$$

Also stated as:

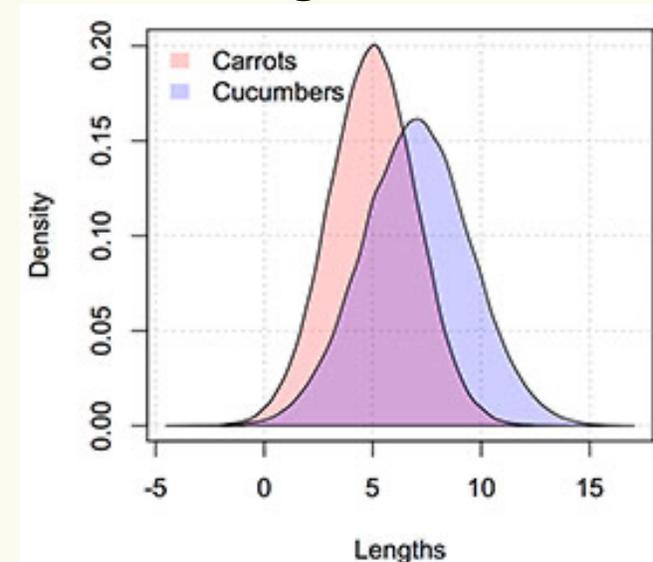
- $\lim_{n \rightarrow \infty} Y_n \rightarrow \mathcal{N}(0,1)$
- $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_i \rightarrow \mathcal{N}\left(\mu, \frac{\sigma^2}{n}\right)$  for  $\mu = \mathbb{E}[X_i]$  and  $\sigma^2 = \text{Var}(X_i)$

# CLT → Normal Distribution EVERYWHERE

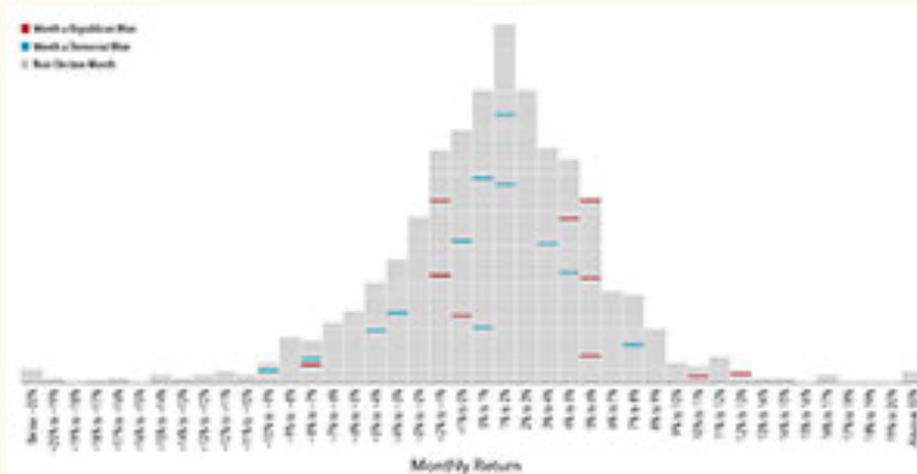
## Neuron Activity



## Vegetables



## S&P 500 Returns after Elections



Examples from:  
<https://galtonboard.com/probabilityexamplesinlife>