

**CSE 312**

# **Foundations of Computing II**

**Lecture 15: Expectation & Variance of Continuous RVs  
Exponential and Normal Distributions**

## Announcements

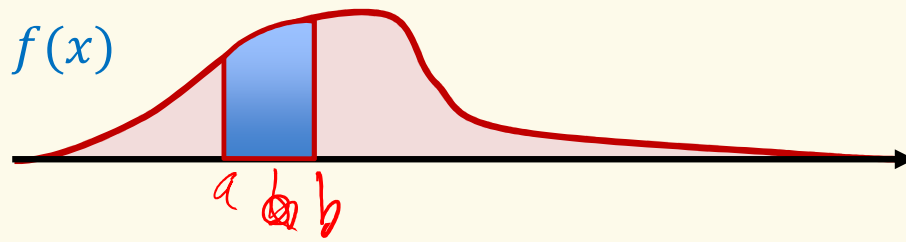
- See EdStem posts related to next week's midterm on May 4 in class:
  - Midterm General Information ✓
  - Midterm Review (including Practice Midterm)
  - Practice Midterm and other Solutions
- Midterm Q&A session next Tuesday 4pm on Zoom

# Review – Continuous RVs

## Probability Density Function (PDF).

$f: \mathbb{R} \rightarrow \mathbb{R}$  s.t.

- $f(x) \geq 0$  for all  $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x) dx = 1$



Density  $\neq$  Probability !

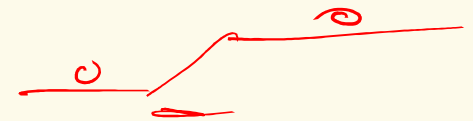
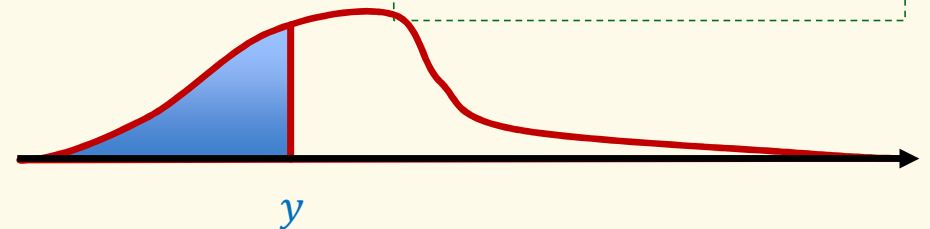


$$\begin{aligned} P(X \in [a, b]) &= \int_a^b f_X(x) dx \\ &= F_X(b) - F_X(a) \end{aligned}$$

## Cumulative Distribution Function (CDF).

$$F(y) = \int_{-\infty}^y f(x) dx$$

**Theorem.**  $f(x) = \frac{dF(x)}{dx}$



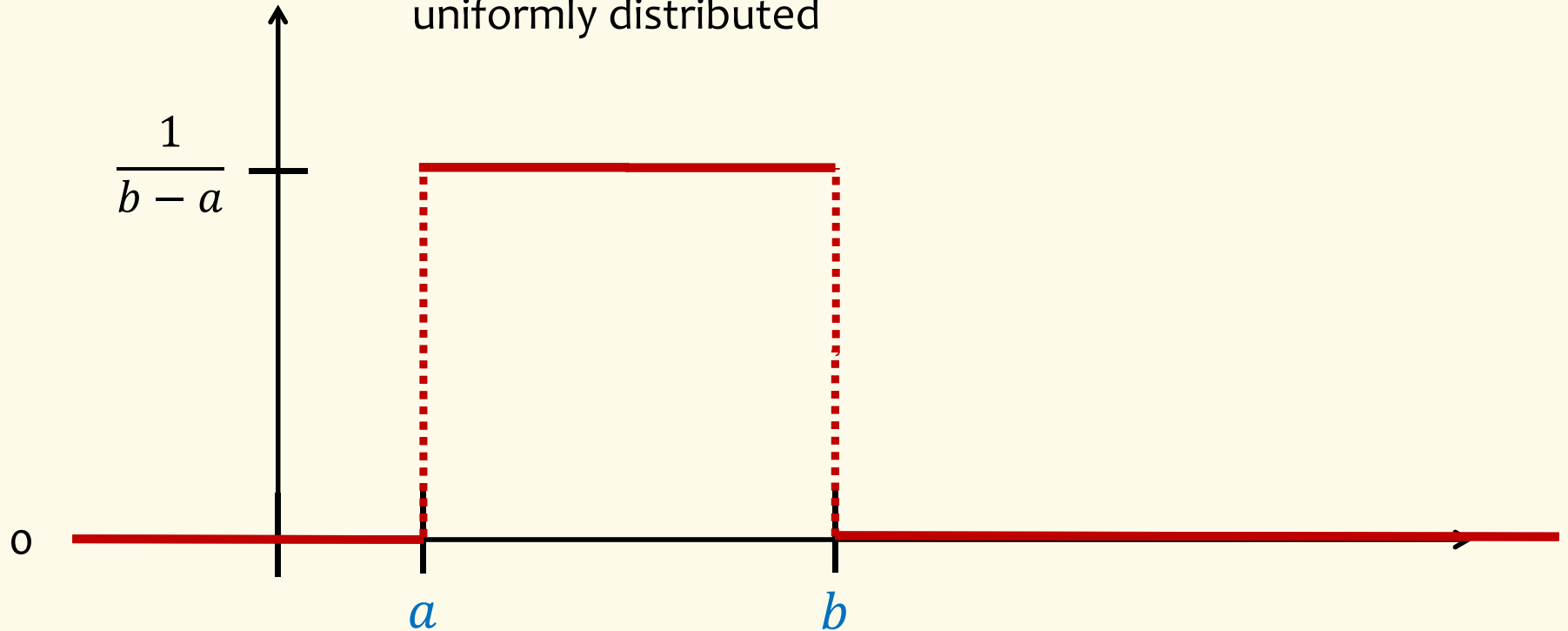
$$F_X(y) = P(X \leq y)$$

## Review: Uniform Distribution

$$X \sim \text{Unif}(a, b)$$

We also say that  $X$  follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



## Review: From Discrete to Continuous

	<b>Discrete</b>	<b>Continuous</b>
<b>PMF/PDF</b>	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
<b>CDF</b>	$F_X(x) = \sum_{t \leq x} p_X(t)$	$F_X(x) = \int_{-\infty}^x f_X(t) dt$
<b>Normalization</b>	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x) dx = 1$
<b>Expectation</b>	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$

## Expectation of a Continuous RV

**Definition.** The **expected value** of a continuous RV  $X$  is defined as

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} \underbrace{f_X(x)} \cdot x \, dx$$

**Fact.**  $\mathbb{E}[aX + bY + c] = a\mathbb{E}[X] + b\mathbb{E}[Y] + c$

Proofs follow same ideas as discrete case

**Definition.** The **variance** of a continuous RV  $X$  is defined as

$$\text{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}[X])^2 \, dx = \underbrace{\mathbb{E}[X^2]} - \underbrace{\mathbb{E}[X]^2}$$

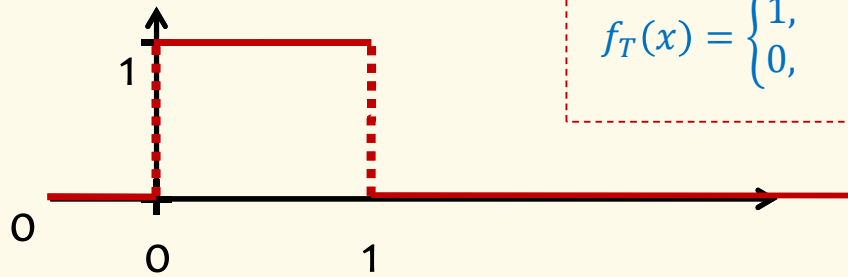
$\mathbb{E}[(x - \mathbb{E}[X])^2]$

# Agenda

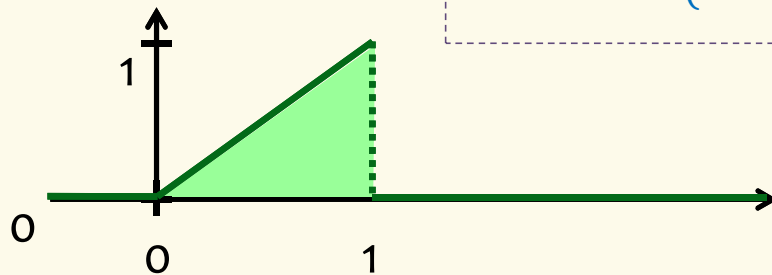
- Uniform Distribution ◀
- Exponential Distribution
- Normal Distribution

# Expectation of a Continuous RV

**Example.**  $T \sim \text{Unif}(0,1)$



$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$f_T(x) \cdot x = \begin{cases} x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

0.5

**Definition.**

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} \underbrace{f_X(x)} \cdot x \, dx$$

$$\mathbb{E}[T] = \underbrace{\frac{1}{2} 1^2}_{\text{Area of triangle}} = \frac{1}{2}$$

Area of triangle



## Uniform Density – Expectation

$X \sim \text{Unif}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} \mathbb{E}[X] &= \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx \\ &= \frac{1}{b-a} \int_a^b x \, dx = \frac{1}{b-a} \left( \frac{x^2}{2} \right) \Big|_a^b = \frac{1}{b-a} \left( \frac{b^2 - a^2}{2} \right) \\ &= \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2} \end{aligned}$$

*Handwritten notes:*  $\frac{1}{2}x^2$  (with a downward arrow),  $(b-a)(b+a)$  (with a double vertical line), and a red circle around  $\frac{a+b}{2}$ .

## Uniform Density – Variance

$X \sim \text{Unif}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\begin{aligned} \mathbb{E}[X^2] &= \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 dx \\ &= \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{b-a} \left( \frac{x^3}{3} \right) \Big|_a^b = \frac{b^3 - a^3}{3(b-a)} \\ &= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3} \end{aligned}$$

*Handwritten notes:*  
- A red underline is under  $\mathbb{E}[X^2]$ .  
- A red arrow points from  $x^2$  to  $\frac{1}{3}x^3$ .  
- A red arrow points from  $\frac{x^3}{3}$  to the  $\frac{1}{3}$  in the denominator of the next step.  
- A red arrow points from  $\frac{x^3}{3}$  to the  $(b-a)$  in the denominator of the next step.  
- A red arrow points from  $\frac{x^3}{3}$  to the  $(b^2 + ab + a^2)$  in the numerator of the next step.  
- A red wavy line is under the final denominator  $3$ .

## Uniform Density – Variance

$$\mathbb{E}[X^2] = \frac{b^2 + ab + a^2}{3} \quad \mathbb{E}[X] = \frac{a + b}{2}$$

$$X \sim \text{Unif}(a, b)$$

$$\mathbb{E}[X]^2 = \left(\frac{a+b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{4}$$

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

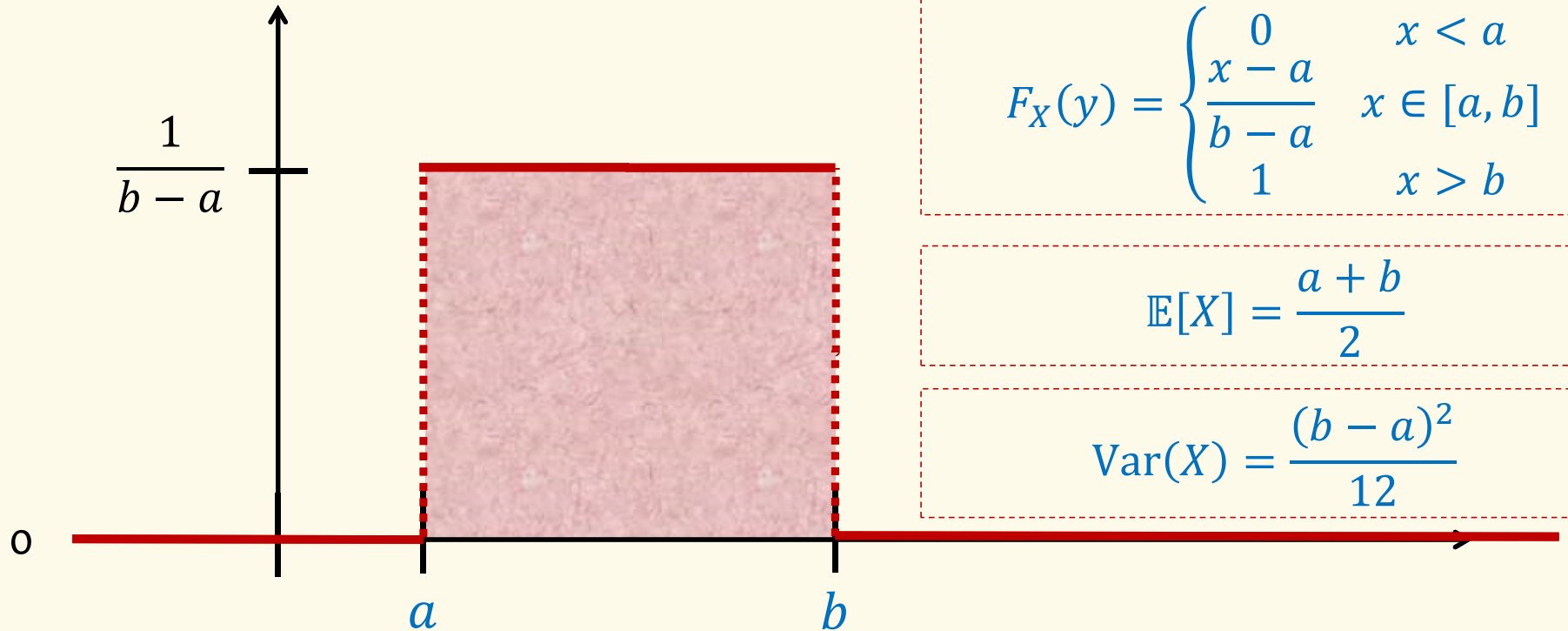
$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b - a)^2}{12}$$

## Uniform Distribution Summary

$X \sim \text{Unif}(a, b)$



$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}[X] = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$

# Agenda

- Uniform Distribution
- Exponential Distribution ◀
- Normal Distribution

## Exponential Density

Assume expected # of occurrences of an event per unit of time is  $\lambda$  (independently)

- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER
- Rate of radioactive decay

**Numbers of occurrences of event:** Poisson distribution

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!} \quad (\text{Discrete})$$

**How long to wait until next event?** Exponential density!

Let's define it and then derive it!

## Exponential Density - Warmup

$$X \sim \text{Poi}(\lambda) \Rightarrow P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

Assume expected # of occurrences of an event per unit of time is  $\lambda$  (independently)

What is the distribution of  $Z = \#$  occurrences of event per  $t$  units of time?

$$\mathbb{E}[Z] = t\lambda$$

$Z$  is independent over disjoint intervals

$$\text{so } Z \sim \text{Poi}(t\lambda)$$

$$X \sim \text{Poi}(\lambda) \Rightarrow P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

## The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is  $\lambda$  (independently)

**Numbers of occurrences of event:** Poisson distribution  $\leftarrow$

**How long to wait until next event?** Exponential density!

- The exponential RV has range  $[0, \infty]$ , unlike Poisson with range  $\{0, 1, 2, \dots\}$   $\mathbb{R}_{\geq 0}$   $\mathbb{N}$   $P(Y \leq t)$
- Let  $Y \sim \text{Exp}(\lambda)$  be the time till the first event. We will compute  $F_Y(t)$  and  $f_Y(t)$
- Let  $Z \sim \text{Poi}(t\lambda)$  be the # of events in the first  $t$  units of time, for  $t \geq 0$ .
- $P(Y > t) = P(\text{no event in the first } t \text{ units}) = P(Z = 0) = e^{-t\lambda} \frac{(t\lambda)^0}{0!} = e^{-t\lambda}$
- $F_Y(t) = P(Y \leq t) = 1 - P(Y > t) = 1 - e^{-t\lambda}$
- $f_Y(t) = \frac{d}{dt} F_Y(t) = \lambda e^{-t\lambda}$



$$P(X > t) = e^{-t\lambda}$$

## Exponential Distribution

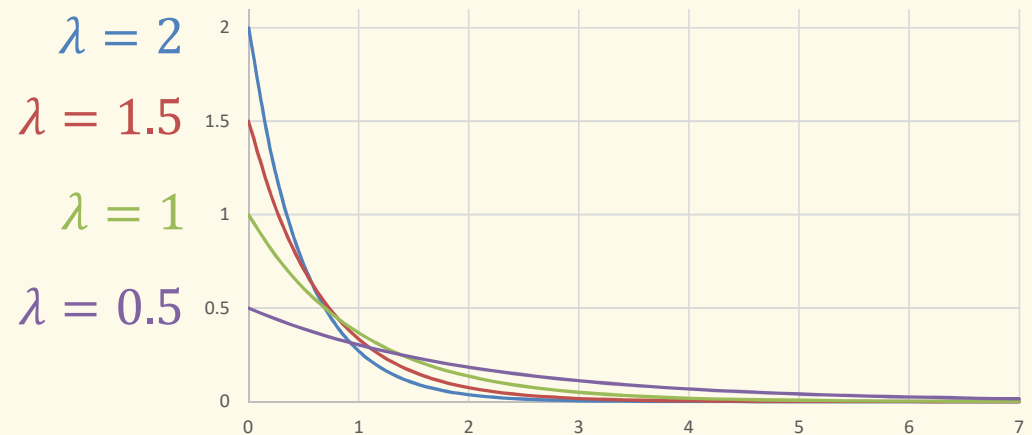
**Definition.** An **exponential random variable**  $X$  with parameter  $\lambda \geq 0$  is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write  $X \sim \text{Exp}(\lambda)$  and say  $X$  that follows the exponential distribution.

CDF: For  $y \geq 0$ ,

$$F_X(y) = 1 - e^{-\lambda y}$$



## Expectation

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx \\ &= \int_0^{+\infty} \lambda e^{-\lambda x} \cdot x \, dx \\ &= \left( -\left(x + \frac{1}{\lambda}\right) e^{-\lambda x} \right) \Big|_0^{\infty} = \frac{1}{\lambda}\end{aligned}$$

Somewhat complex calculation  
use integral by parts

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

$$P(X > t) = e^{-t\lambda}$$

$$\mathbb{E}[X] = \frac{1}{\lambda}$$

$$\text{Var}(X) = \frac{1}{\lambda^2}$$



## Memorylessness

**Definition.** A random variable is **memoryless** if for all  $s, t > 0$ ,

$$P(X > s + t \mid X > s) = P(X > t).$$

**Fact.**  $X \sim \text{Exp}(\lambda)$  is memoryless.

Assuming an exponential distribution, if you've waited  $s$  minutes, The probability of waiting  $t$  more is exactly same as when  $s = 0$ .

## Memorylessness of Exponential

$$P(X > t) = e^{-\lambda t}$$

Proof that assuming exp distr, if you've waited  $s$  minutes, prob of waiting  $t$  more is exactly same as when  $s = 0$

**Fact.**  $X \sim \text{Exp}(\lambda)$  is memoryless.

**Proof.**

$$\begin{aligned} P(X > s + t | X > s) &= \frac{P(\{X > s + t\} \cap \{X > s\})}{P(X > s)} \\ &= \frac{P(X > s + t)}{P(X > s)} \\ &= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t) \end{aligned}$$

*Handwritten notes:*  $X > s + t$  (circled in red),  $1 - F_\lambda(s+t)$ ,  $1 - F(s)$

The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous)

## Example

$$\lambda = \frac{1}{10}$$

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?

$$T \sim \text{Exp}\left(\frac{1}{10}\right)$$

$$P(10 \leq T \leq 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} dx \quad dy$$

$$y = \frac{x}{10} \text{ so } dy = \frac{dx}{10}$$

$$P(10 \leq T \leq 20) = \int_1^2 e^{-y} dy = -e^{-y} \Big|_1^2 = e^{-1} - e^{-2}$$

## Example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins?

$$T \sim \text{Exp}\left(\frac{1}{10}\right)$$

$$\text{so } F_T(t) = 1 - e^{-\frac{t}{10}}$$

$$\begin{aligned} P(10 \leq T \leq 20) &= F_T(20) - F_T(10) \\ &= 1 - e^{-\frac{20}{10}} - \left(1 - e^{-\frac{10}{10}}\right) = e^{-1} - e^{-2} \end{aligned}$$

# Agenda

- Uniform Distribution
- Exponential Distribution
- Normal Distribution ◀

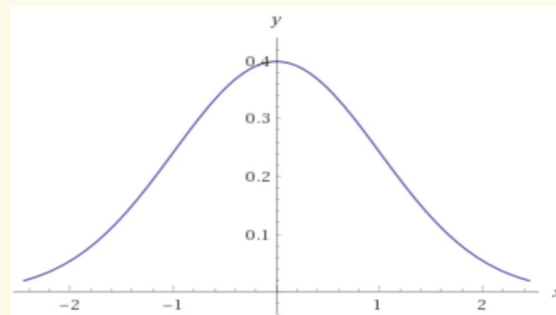


# The Normal Distribution

**Definition.** A **Gaussian (or normal)** random variable with parameters  $\mu \in \mathbb{R}$  and  $\sigma \geq 0$  has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

We say that  $X$  follows the Normal Distribution, and write  $X \sim \mathcal{N}(\mu, \sigma^2)$ .



$\mathcal{N}(0, 1)$ .



Carl Friedrich  
Gauss

## The Normal Distribution

**Definition.** A **Gaussian (or normal) random variable** with parameters  $\mu \in \mathbb{R}$  and  $\sigma \geq 0$  has density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

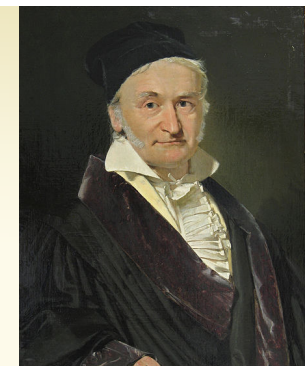
We say that  $X$  follows the Normal Distribution, and write  $X \sim \mathcal{N}(\mu, \sigma^2)$ .

**Fact.** If  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then  $\mathbb{E}[X] = \mu$ , and  $\text{Var}(X) = \sigma^2$

Proof of expectation is easy because density curve is symmetric around  $\mu$ ,

$f_X(\mu - x) = f_X(\mu + x)$ , but proof for variance requires integration of  $e^{-x^2/2}$

We will see next time why the normal distribution is (in some sense) the most important distribution.



Carl Friedrich  
Gauss

$\frac{1}{\sqrt{2\pi}}$

# The Normal Distribution

Aka a “Bell Curve” (imprecise name)

