Lecture 14: Continuous RV
Announcements

• PSet 4 due today
• PSet 3 returned yesterday
• Midterm general info is posted on Ed
• Practice midterm is posted
  – Has format you will see, including 2-page “cheat sheet”.
  – Other practice materials linked also
• Midterm Q&A session next Tuesday 4pm on Zoom
Often we want to model experiments where the outcome is not discrete.
Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

- \( T \) = time of lightning strike

- Every time within \([0,1]\) is equally likely
  - Time measured with infinitesimal precision.

The outcome space is not discrete

\[ T = 0.71237131931129576 \ldots \]
Lightning strikes a pole within a one-minute time frame

• $T =$ time of lightning strike
• Every point in time within $[0,1]$ is equally likely

\[ P(T \geq 0.5) = \frac{1}{2} \]
Lightning strikes a pole within a one-minute time frame

- $T$ = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

\[ P(0.2 \leq T \leq 0.5) = 0.5 - 0.2 = 0.3 \]
Lightning strikes a pole within a one-minute time frame

- $T =$ time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$P(T = 0.5) = 0$$
Bottom line

• This gives rise to a different type of random variable
• \( P(T = x) = 0 \) for all \( x \in [0,1] \)
• Yet, somehow we want
  – \( P(T \in [0,1]) = 1 \)
  – \( P(T \in [a, b]) = b - a \)
  – ...
• How do we model the behavior of \( T \)?

  First try: A discrete approximation
Recall: Cumulative Distribution Function (CDF)

Probability Mass Function (PMF)

$p_X$

Cumulative Distribution Function (CDF)

$F_X$
A Discrete Approximation

**Probability Mass Function (PMF)**

- \( p_X \)

**Cumulative Distribution Function (CDF)**

- \( F_X \)
Definition. A **continuous random variable** \( X \) is defined by a probability density function (PDF) \( f_X : \mathbb{R} \rightarrow \mathbb{R} \), such that

**Non-negativity:** \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)
Probability Density Function - Intuition

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

Normalization: \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)
Probability Density Function - Intuition

**Non-negativity:** $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

**Normalization:** $\int_{-\infty}^{+\infty} f_X(x) \, dx = 1$

$$P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx$$
Probability Density Function - Intuition

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

Normalization: \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

\[
P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx
\]

\[
P(X = y) = P(y \leq X \leq y) = \int_{y}^{y} f_X(x) \, dx = 0
\]

Density ≠ Probability

\( f_X(y) \neq 0 \quad P(X = y) = 0 \)
Probability Density Function - Intuition

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

Normalization: \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

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P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx
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P(X = y) = P(y \leq X \leq y) = \int_{y}^{y} f_X(x) \, dx = 0
\]

\[
P(X \approx y) \approx P \left( y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2} \right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x) \, dx \approx \epsilon f_X(y)
\]

What \( f_X(x) \) measures: The local rate at which probability accumulates

\[
y - \frac{\epsilon}{2} \quad y \quad y + \frac{\epsilon}{2}
\]
Probability Density Function - Intuition

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x) \, dx = 1$

$P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx$

$P(X = y) = P(y \leq X \leq y) = \int_{y}^{y} f_X(x) \, dx = 0$

$P(X \approx y) \approx P \left( y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2} \right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x) \, dx \approx \epsilon f_X(y)$

$P(X \approx y) \approx \epsilon f_X(y) \approx \frac{f_X(y)}{f_X(z)}$
**Definition.** A **continuous random variable** $X$ is defined by a probability density function (PDF) $f_X: \mathbb{R} \to \mathbb{R}$, such that

**Non-negativity:** $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

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$P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx$

$P(X = y) = P(y \leq X \leq y) = \int_{y}^{y} f_X(x) \, dx = 0$

$P(X \approx y) \approx P \left( y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2} \right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x) \, dx \approx \epsilon f_X(y)$

$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$
PDF of Uniform RV

X ∼ Unif(0,1)

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

Normalization: \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

\[
f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}
\]

\[
\int_{-\infty}^{+\infty} f_X(x) \, dx = \int_{0}^{1} f_X(x) \, dx = 1 \cdot 1 = 1
\]
Probability of Event

\[ X \sim \text{Unif}(0,1) \]

Non-negativity: \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

Normalization: \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

\[ P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx \]

1. If \( 0 \leq a \) and \( a \leq b \leq 1 \)
   \[ P(a \leq X \leq b) = b - a \]
2. If \( a < 0 \) and \( 0 \leq b \leq 1 \)
   \[ P(a \leq X \leq b) = b \]
3. If \( a \geq 0 \) and \( b > 1 \)
   \[ P(a \leq X \leq b) = b - a \]
4. If \( a < 0 \) and \( b > 1 \)
   \[ P(a \leq X \leq b) = 1 \]

Poll: pollev/paulbeame028

A. All of them are correct
B. Only 1, 2, 4 are right
C. Only 1 is right
D. Only 1 and 2 are right
**Probability of Event**

\( X \sim \text{Unif}(0,1) \)

\[
f_X(x) = \begin{cases} 
1, & x \in [0,1] \\
0, & x \notin [0,1] 
\end{cases}
\]

**Non-negativity:** \( f_X(x) \geq 0 \) for all \( x \in \mathbb{R} \)

**Normalization:** \( \int_{-\infty}^{+\infty} f_X(x) \, dx = 1 \)

\[
P(a \leq X \leq b) = \int_{a}^{b} f_X(x) \, dx
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\[
P(X = y) = P(y \leq X \leq y) = \int_{y}^{y} f_X(x) \, dx = 0
\]

\[
P(X \approx y) \approx \epsilon f_X(y) = \epsilon
\]

\[
\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}
\]
PDF of Uniform RV

\[ X \sim \text{Unif}(0,0.5) \]

\[ f_X(x) = \begin{cases} 
2, & x \in [0,0.5] \\
0, & x \notin [0,0.5] 
\end{cases} \]

\[ \int_{-\infty}^{+\infty} f_X(x) \, dx = \int_{0}^{1} f_X(x) \, dx = 2 \cdot 0.5 = 1 \]

Probability on \([0,0.5]\) accumulates at twice the rate compared to \(\text{Unif}(0,1)\)
Uniform Distribution

\[ X \sim \text{Unif}(a, b) \]

\[ f_X(x) = \begin{cases} 
\frac{1}{b-a} & x \in [a, b] \\
0 & \text{else} 
\end{cases} \]

\[ \int_{-\infty}^{+\infty} f_X(x) \, dx = (b-a) \frac{1}{b-a} = 1 \]
**Example.** $T \sim \text{Unif}(0,1)$

**Probability Density Function**

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

**Cumulative Distribution Function**

$$F_T(x) = P(T \leq x) = \begin{cases} 0, & x \leq 0 \\ ? & 0 \leq x \leq 1 \\ 1, & 1 \leq x \end{cases}$$
Cumulative Distribution Function

**Definition.** The **cumulative distribution function** (cdf) of $X$ is

$$F_X(a) = P(X \leq a) = \int_{-\infty}^{a} f_X(x) \, dx$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx} F_X(x)$

Therefore: $P(X \in [a, b]) = F_X(b) - F_X(a)$

$F_X$ is monotone increasing, since $f_X(x) \geq 0$. That is $F_X(c) \leq F_X(d)$ for $c \leq d$

$$\lim_{a \to -\infty} F_X(a) = P(X \leq -\infty) = 0 \quad \lim_{a \to +\infty} F_X(a) = P(X \leq +\infty) = 1$$
## From Discrete to Continuous

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<thead>
<tr>
<th></th>
<th><strong>Discrete</strong></th>
<th><strong>Continuous</strong></th>
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</thead>
<tbody>
<tr>
<td><strong>PMF/PDF</strong></td>
<td>$p_X(x) = P(X = x)$</td>
<td>$f_X(x) \neq P(X = x) = 0$</td>
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<tr>
<td><strong>CDF</strong></td>
<td>$F_X(x) = \sum_{t \leq x} p_X(t)$</td>
<td>$F_X(x) = \int_{-\infty}^{x} f_X(t) , dt$</td>
</tr>
<tr>
<td><strong>Normalization</strong></td>
<td>$\sum_x p_X(x) = 1$</td>
<td>$\int_{-\infty}^{\infty} f_X(x) , dx = 1$</td>
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<tr>
<td><strong>Expectation</strong></td>
<td>$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$</td>
<td>$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) , dx$</td>
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