

CSE 312

Foundations of Computing II

Lecture 13: Poisson Distribution

Announcements

- (temporary) office hours + class change
 - Today's office hours moved to 10:30-11:30
 - Friday's office hour is canceled
 - Drop me e-mail to schedule zoom meeting
 - Prof. Beame will give Section A classes on Wed + Fri this week
- Midterm info is posted
 - Q&A session next Tuesday 4pm on Zoom
 - Practice midterm + other practice materials posted this Wednesday

Zoo of Random Variables

$X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$
$$E[X] = \frac{a + b}{2}$$
$$\text{Var}(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$
$$E[X] = p$$
$$\text{Var}(X) = p(1 - p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$
$$E[X] = np$$
$$\text{Var}(X) = np(1 - p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$
$$E[X] = \frac{1}{p}$$
$$\text{Var}(X) = \frac{1 - p}{p^2}$$

$X \sim \text{NegBin}(r, p)$

$$P(X = k) = \binom{k-1}{r-1} p^r (1 - p)^{k-r}$$
$$E[X] = \frac{r}{p}$$
$$\text{Var}(X) = \frac{r(1 - p)}{p^2}$$

$X \sim \text{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k} \binom{N-K}{n-k}}{\binom{N}{n}}$$
$$E[X] = n \frac{K}{N}$$
$$\text{Var}(X) = n \frac{K(N-K)(N-n)}{N^2(N-1)}$$

Agenda

- Poisson Distribution 
- Approximate Binomial distribution using Poisson distribution

Preview: Poisson

Model: # events that occur in an hour

- Expect to see 3 events per hour (but will be random)
- The expected number of events in t hours, is $3t$
- Occurrence of events on disjoint time intervals is independent

Example – Modelling car arrivals at an intersection

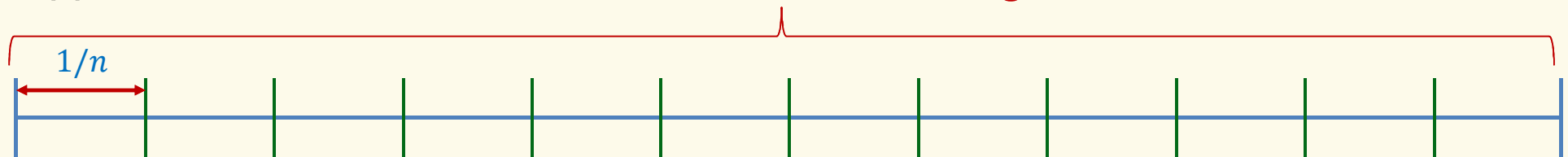
X = # of cars passing through a light in 1 hour

Example – Model the process of cars passing through a light in 1 hour

$X = \#$ cars passing through a light in 1 hour. $\mathbb{E}[X] = 3$

Assume: Occurrence of events on disjoint time intervals is independent

Approximation idea: Divide hour into n intervals of length $1/n$



This gives us n independent intervals

Assume at most one car per interval

$p =$ probability car arrives in an interval

What should p be?

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A. $3/n$

B. $3n$

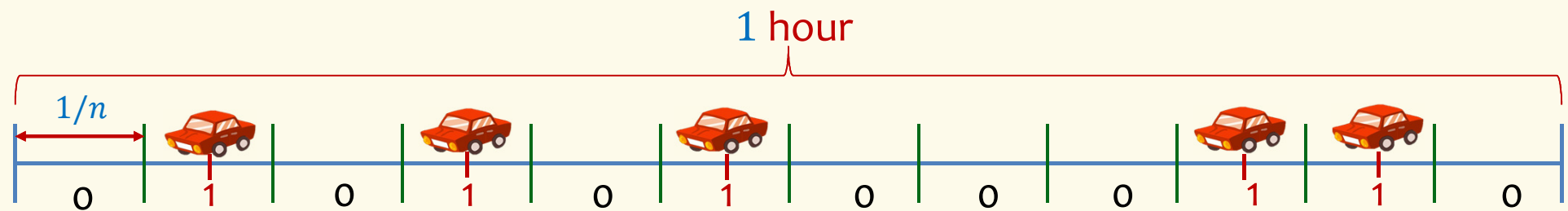
C. 3

D. $3/60$

Example – Model the process of cars passing through a light in 1 hour

X = # cars passing through a light in 1 hour. Disjoint time intervals are independent.

Know: $\mathbb{E}[X] = \lambda$ for some given $\lambda > 0$



Discrete version: n intervals, each of length $1/n$.

In each interval, there is a car with probability $p = \lambda/n$ (assume ≤ 1 car can pass by)

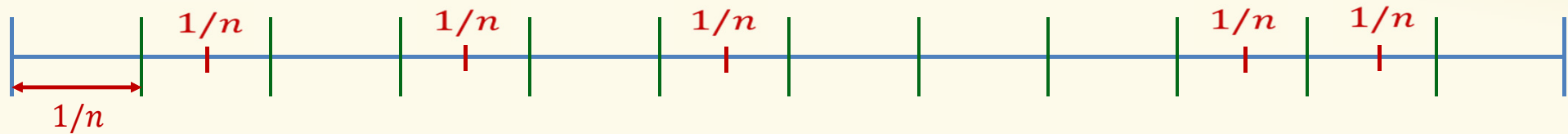
Each interval is Bernoulli: $X_i = 1$ if car in i^{th} interval (0 otherwise). $P(X_i = 1) = \lambda/n$

$$X = \sum_{i=1}^n X_i \quad X \sim \text{Bin}(n, p) \quad P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$$

indeed! $\mathbb{E}[X] = pn = \lambda$

Don't like discretization

X is binomial $P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i}$



We want now $n \rightarrow \infty$

$$P(X = i) = \binom{n}{i} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} = \underbrace{\frac{n!}{(n-i)! n^i}}_{\rightarrow 1} \frac{\lambda^i}{i!} \underbrace{\left(1 - \frac{\lambda}{n}\right)^n}_{\rightarrow e^{-\lambda}} \underbrace{\left(1 - \frac{\lambda}{n}\right)^{-i}}_{\rightarrow 1}$$
$$\rightarrow P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Poisson Distribution

- Suppose “events” happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. with parameter λ (denoted $X \sim \text{Poi}(\lambda)$) and has distribution (PMF):

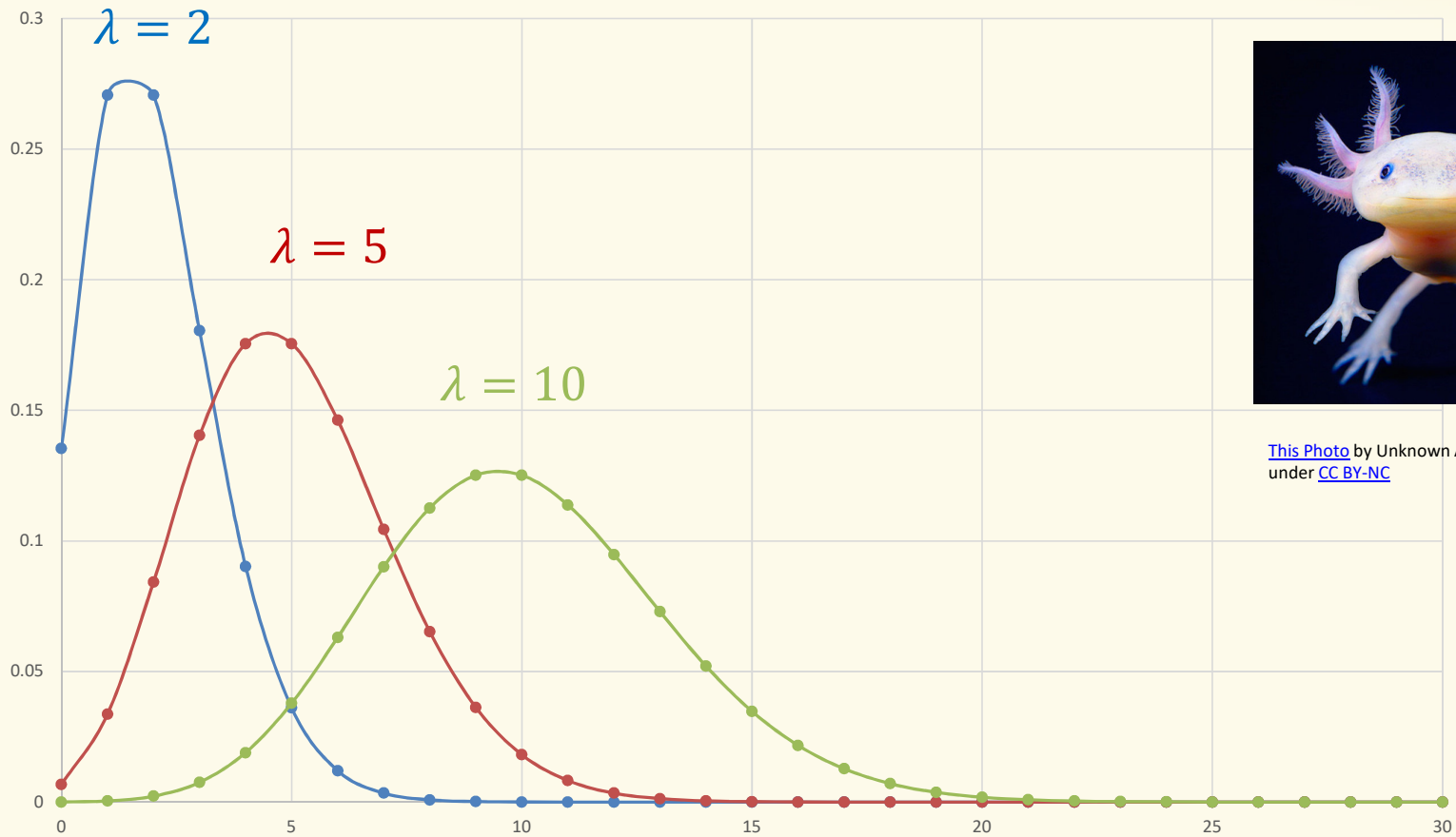
$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Several examples of “Poisson processes”:

- # of cars passing through a traffic light in 1 hour
 - # of requests to web servers in an hour
 - # of photons hitting a light detector in a given interval
 - # of patients arriving to ER within an hour
- Assume fixed average rate

Probability Mass Function

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



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Validity of Distribution

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} P(X = i) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = e^{-\lambda} \underbrace{\sum_{i=0}^{\infty} \frac{\lambda^i}{i!}}_{e^{\lambda}} = e^{-\lambda} e^{\lambda} = 1$$

Fact (Taylor series expansion):

$$e^x = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

Expectation

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ , then

$$\mathbb{E}[X] = \lambda$$

Proof.

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=0}^{\infty} P(X = i) \cdot i = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} \\ &= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \\ &= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda\end{aligned}$$

$= 1$ (see prior slides!)

Variance

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ , then $\text{Var}(X) = \lambda$

Proof.

$$\begin{aligned}\mathbb{E}[X^2] &= \sum_{i=0}^{\infty} P(X = i) \cdot i^2 = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i \\ &= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1) \\ &= \lambda \left[\underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}[X] = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda\end{aligned}$$

Similar to the previous proof
Verify offline.



$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$



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Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0, 1, 2, 3 \dots$,

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



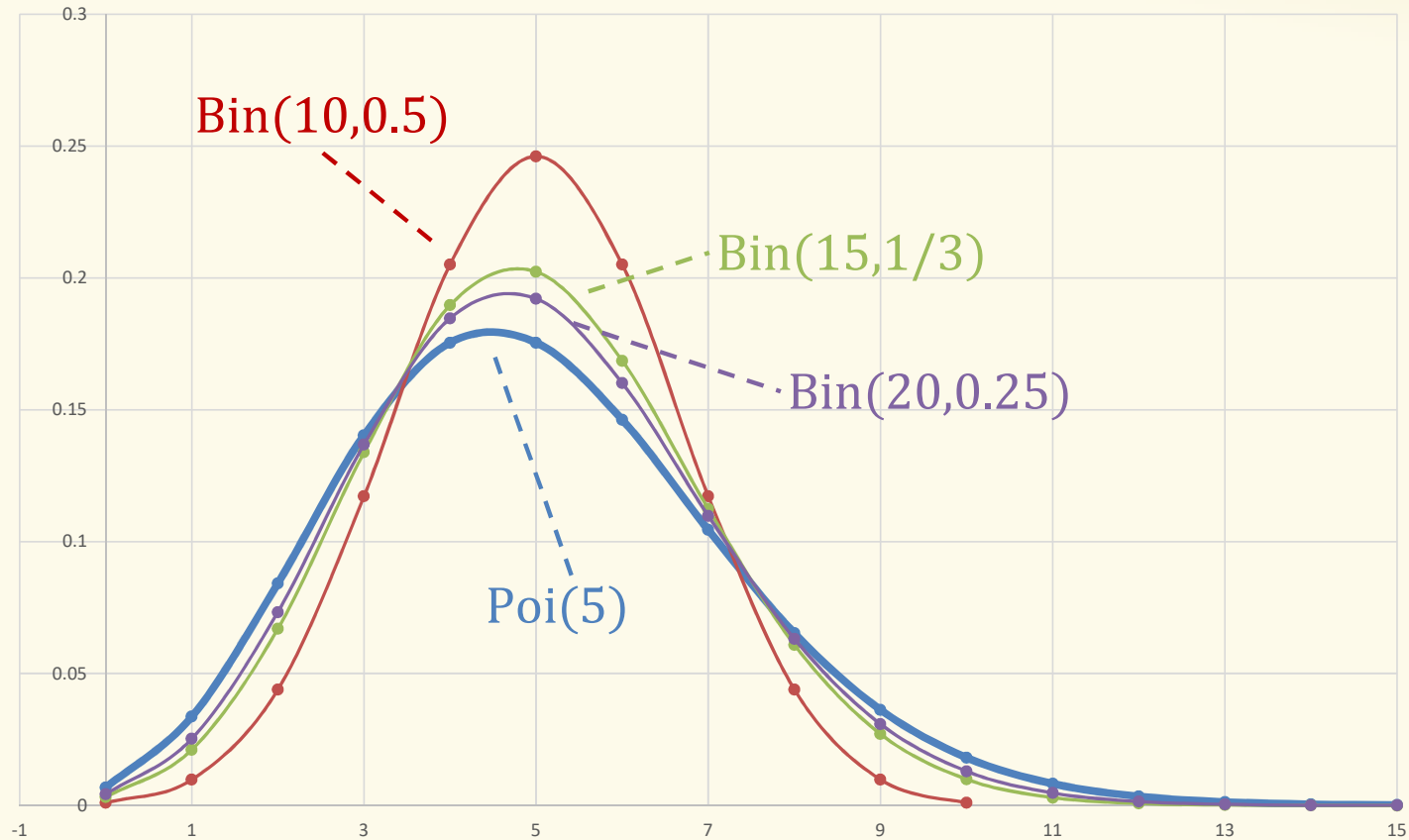
Poisson approximates binomial when:

n is very large, p is very small, and $\lambda = np$ is “moderate”
e.g. ($n > 20$ and $p < 0.05$), ($n > 100$ and $p < 0.1$)

Formally, Binomial approaches Poisson in the limit as $n \rightarrow \infty$ (equivalently, $p \rightarrow 0$) while holding $np = \lambda$

Probability Mass Function – Convergence of Binomials

$$\lambda = 5$$
$$p = \frac{5}{n}$$
$$n = 10, 15, 20$$



as $n \rightarrow \infty$, $\text{Binomial}(n, p = \lambda/n) \rightarrow \text{poi}(\lambda)$


From Binomial to Poisson

$$X \sim \text{Bin}(n, p)$$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np$$

$$\text{Var}(X) = np(1 - p)$$

$$\begin{aligned} n &\rightarrow \infty \\ np &= \lambda \\ p &= \frac{\lambda}{n} \rightarrow 0 \end{aligned}$$


$$X \sim \text{Poi}(\lambda)$$

$$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$E[X] = \lambda$$

$$\text{Var}(X) = \lambda$$

Example -- Approximate Binomial Using Poisson

Consider sending bit string over a network

- Send bit string of length $n = 10^4$
- Probability of (independent) bit corruption is $p = 10^{-6}$

What is probability that message arrives uncorrupted?

Using $X \sim \text{Poi}(\lambda = np = 10^4 \cdot 10^{-6} = 0.01)$

$$P(X = 0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-0.01} \cdot \frac{0.01^0}{0!} \approx 0.990049834$$

Using $Y \sim \text{Bin}(10^4, 10^{-6})$

$$P(Y = 0) \approx 0.990049829$$



Sum of Independent Poisson RVs

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = X + Y$. For all $z = 0, 1, 2, 3, \dots$,

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

More generally, let $X_1 \sim \text{Poi}(\lambda_1), \dots, X_n \sim \text{Poi}(\lambda_n)$ such that $\lambda = \sum_i \lambda_i$.

Let $Z = \sum_i X_i$

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

Sum of Independent Poisson RVs

Theorem. Let $X \sim \text{Poi}(\lambda_1)$ and $Y \sim \text{Poi}(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.

Let $Z = X + Y$. For all $z = 0, 1, 2, 3, \dots$,

$$P(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

$P(Z = z) = ?$

1. $P(Z = z) = \sum_{j=0}^z P(X = j, Y = z - j)$
2. $P(Z = z) = \sum_{j=0}^{\infty} P(X = j, Y = z - j)$
3. $P(Z = z) = \sum_{j=0}^z P(Y = z - j | X = j) P(X = j)$
4. $P(Z = z) = \sum_{j=0}^z P(Y = z - j | X = j)$

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- A. All of them are right
- B. The first 3 are right
- C. Only 1 is right
- D. Don't know

Proof

$$P(Z = z) = \sum_{j=0}^k P(X = j, Y = z - j)$$

Law of total probability

$$= \sum_{j=0}^k P(X = j) P(Y = z - j) = \sum_{j=0}^k e^{-\lambda_1} \cdot \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{z-j}}{z-j!}$$

Independence

$$= e^{-\lambda_1 - \lambda_2} \left(\sum_{j=0}^k \frac{1}{j! z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right)$$

$$= e^{-\lambda} \left(\sum_{j=0}^k \frac{z!}{j! z - j!} \cdot \lambda_1^j \lambda_2^{z-j} \right) \frac{1}{z!}$$

$$= e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^z \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^z \cdot \frac{1}{z!}$$

Binomial
Theorem

Poisson Random Variables


Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0, 1, 2, 3 \dots$,

$$P(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

General principle:

- Events happen at an average rate of λ per time unit
- Number of events happening at a time unit X is distributed according to $\text{Poi}(\lambda)$
- Poisson approximates Binomial when n is large, p is small, and np is moderate
- Sum of independent Poisson is still a Poisson

Next

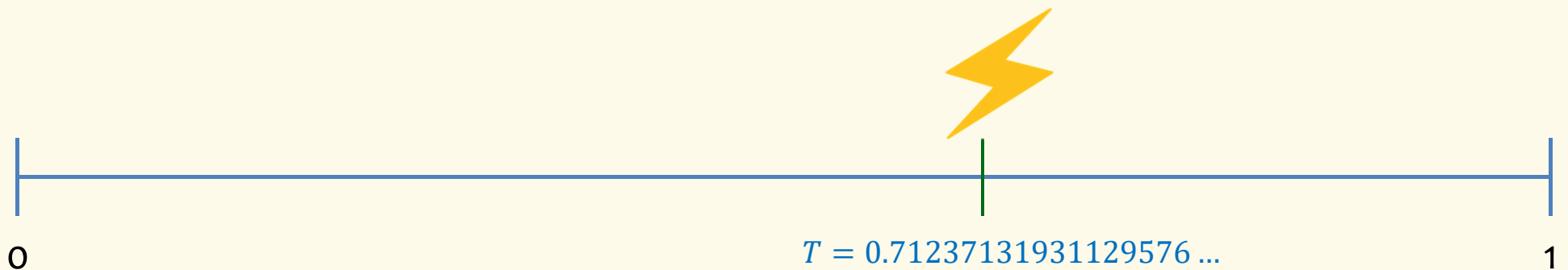
- Continuous Random Variables 
- Probability Density Function
- Cumulative Density Function

Often we want to model experiments where the outcome is not discrete.

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

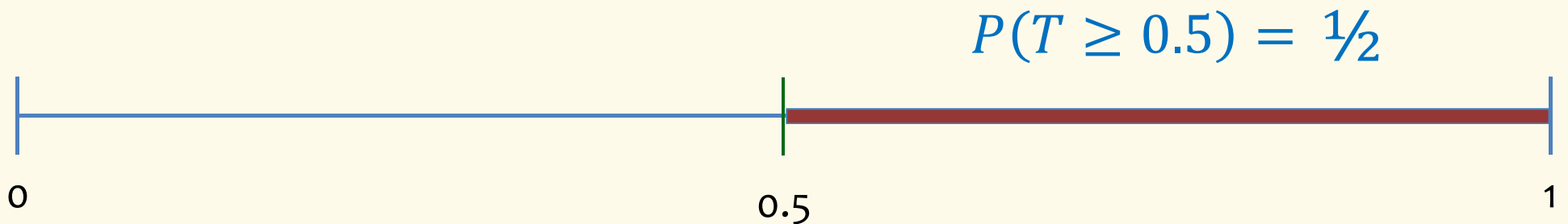
- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
 - Time measured with infinitesimal precision.



The outcome space is not discrete

Lightning strikes a pole within a one-minute time frame

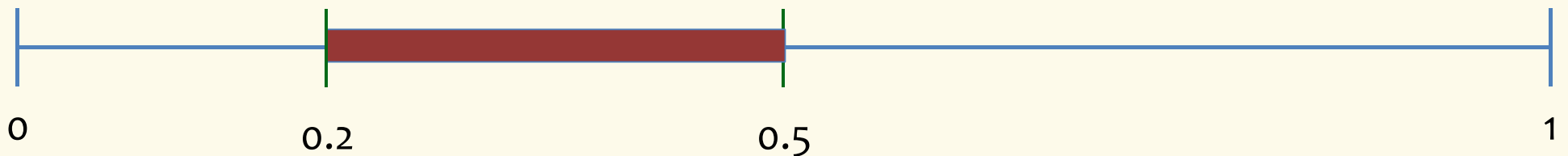
- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



Lightning strikes a pole within a one-minute time frame

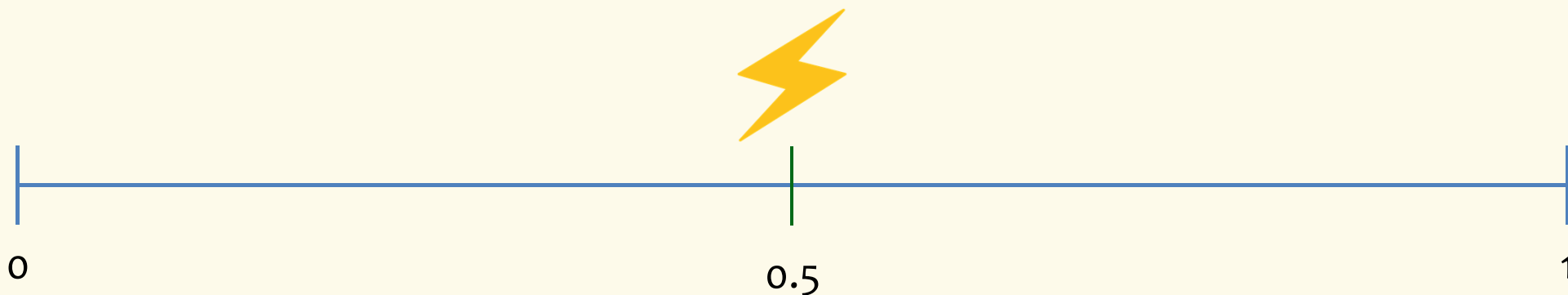
- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely

$$P(0.2 \leq T \leq 0.5) = 0.5 - 0.2 = 0.3$$



Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



$$P(T = 0.5) = 0$$

Bottom line

- This gives rise to a different type of random variable
- $P(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
 - $P(T \in [0,1]) = 1$
 - $P(T \in [a, b]) = b - a$
 - ...
- How do we model the behavior of T ?