CSE 312

Foundations of Computing II

Lecture 6: Conditional Probability and Bayes Theorem

Review Probability

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die :

$$E = \{2, 4, 6\}$$

Review Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair (Ω, P) where:

- Ω is a set called the **sample space.**
- P is the **probability measure,** a function $P: \Omega \to \mathbb{R}$ such that:
  - $P(x) \geq 0$ for all $x \in \Omega$
  - $\sum_{x \in \Omega} P(x) = 1$

Set of possible **elementary outcomes**

$$A \subseteq \Omega:\ P(A) = \sum_{x \in A} P(x)$$

Specify Likelihood (or probability) of each **elementary outcome**

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Review Axioms of Probability

Let (Ω, P) be a probability space. Then, the following properties hold for any two events $E, F \subseteq \Omega$.

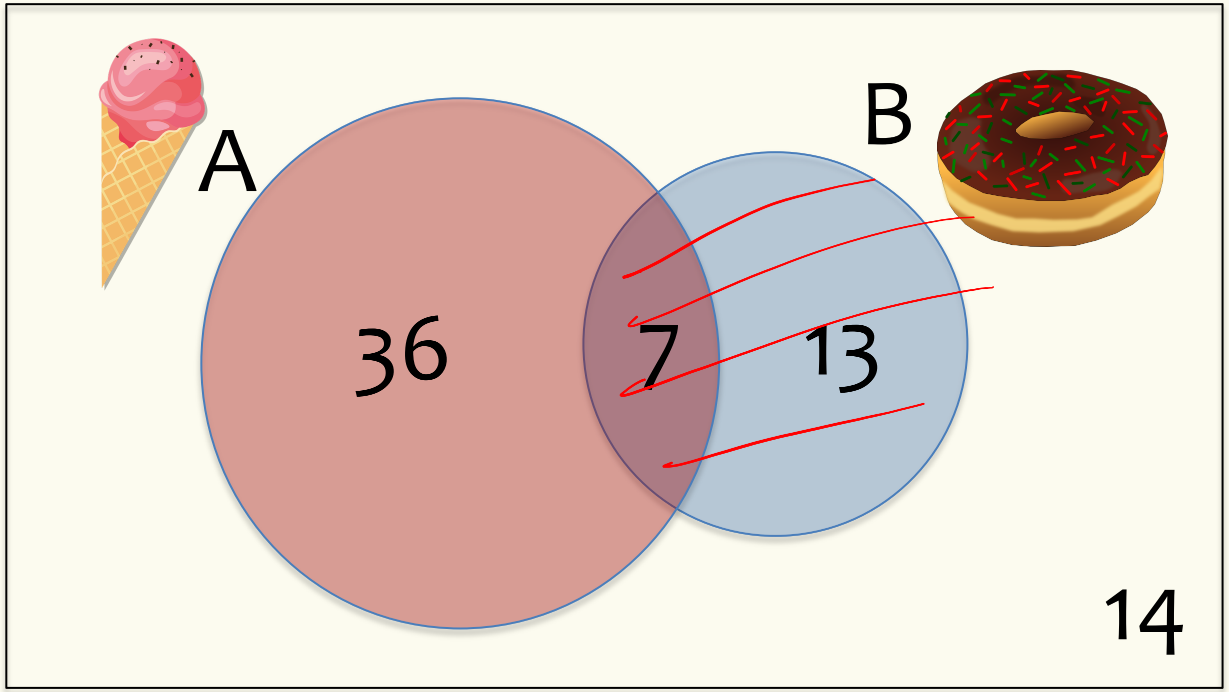
Axiom 1 (Non-negativity): $P(E) \geq 0$.
Axiom 2 (Normalization): $P(\Omega) = 1$.
Axiom 3 (Countable Additivity): If E and F are mutually exclusive,
then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$.
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- More Examples

Conditional Probability (Idea)



What's the probability that someone likes ice cream **given** they like donuts?

Conditional Probability

Definition. The **conditional probability** of event A **given** an event B happened (assuming $P(B) \neq 0$) is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

An equivalent and useful formula is

$$P(A \cap B) = P(A|B)P(B)$$

Conditional Probability Examples

Suppose that you flip a fair coin twice.

What is the probability that both flips are heads given that you have at least one head?

$B \cap O = \{HH\}$

$O = \{HH, HT, TH\}$

$B = \{HH\}$

Let O be the event that at least *one* flip is heads

Let B be the event that *both* flips are heads

Ω

HH	HT
TH	TT

$P(O) = 3/4 \qquad P(B) = 1/4 \qquad P(B \cap O) = 1/4$

$$P(B|O) = \frac{P(B \cap O)}{P(O)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Conditional Probability Examples

$P(H \mid T)$

Suppose that you flip a fair coin twice.

What is the probability that at least one flip is heads given that at least one flip is tails?

$H = \{HT, TH, HH\}$

$T = \{HT, TH, TT\}$

Let H be the event that at least one flip is *heads*

Let T be the event that at least one flip is *tails*

Ω

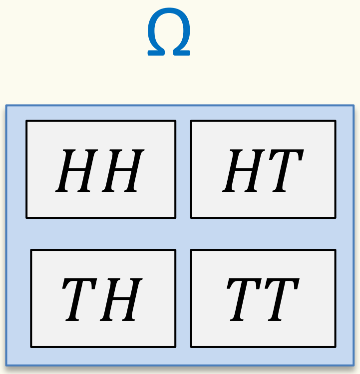
HH	HT
TH	TT

$$P(H \mid T) = \frac{P(H \cap T)}{P(T)} = \frac{P(\{HT, TH\})}{3/4} = \frac{1/2}{3/4} = \frac{1}{2} \cdot \frac{4}{3} = \frac{2}{3}$$

Conditional Probability Examples

Suppose that you flip a fair coin twice.
What is the probability that at least one flip is heads given that at least one flip is tails?

Let H be the event that at least one flip is *heads*
Let T be the event that at least one flip is *tails*



$P(H) = 3/4 \qquad P(T) = 3/4 \qquad P(H \cap T) = 1/2$

$$P(H|T) = \frac{P(H \cap T)}{P(T)} = \frac{1/2}{3/4} = \frac{2}{3}$$

Reversing Conditional Probability

Question: Does $P(A|B) = P(B|A)$?

No!

- Let A be the event you are wet
- Let B be the event you are swimming

$$P(A|B) = 1$$
$$P(B|A) \neq 1$$

Example with Conditional Probability

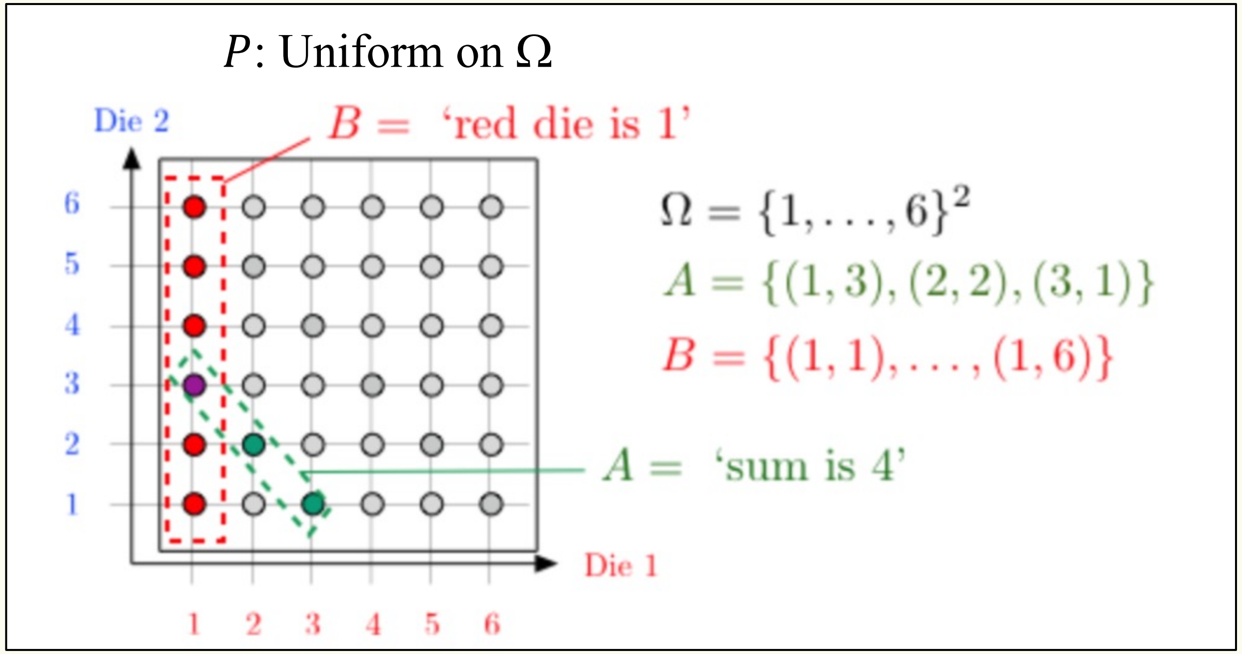
pollev.com/stefanotessaro617

Suppose we toss a red die and a blue die:

both 6 sided and all outcomes equally likely.

What is $P(B)$? What is $P(B|A)$?

	$P(B)$	$P(B\|A)$
a)	1/6	1/6
b)	1/6	1/3
c)	1/6	3/36
d)	1/9	1/3



$P(B) = \frac{6}{36} = \frac{1}{6}$ $\frac{1}{36}$

$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(\{(1,3)\})}{3/36}$

$= 1/3$

Gambler's fallacy

Assume we toss **51** fair coins.
Assume we have seen **50** coins, and they are all "tails".
What are the odds the **51st** coin is "heads"?

A = first 50 coins are "tails"
B = first 50 coins are "tails", 51st coin is "heads"

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{1/2^{51}}{2/2^{51}} = \frac{1}{2}$$

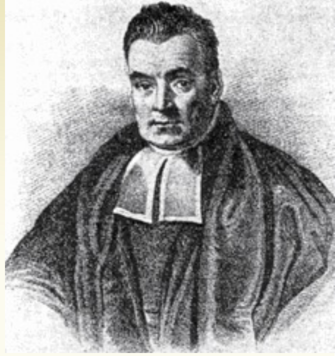
51st coin is independent of outcomes of first 50 tosses!

Gambler's fallacy = Feels like it's time for "heads"!?

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- More Examples

Bayes Theorem



A formula to let us "reverse" the conditional.

Theorem. (Bayes Rule) For events A and B, where $P(A), P(B) > 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$P(A)$ is called the **prior** (our belief without knowing anything)

$P(A|B)$ is called the **posterior** (our belief after learning B)

Bayes Theorem Proof

Claim:

$$P(A), P(B) > 0 \quad \Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

$$P(A \cap B) = P(A) \cdot B(B|A)$$

$$=$$

$$P(B \cap A) = P(B) \cdot P(A|B)$$

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(B)}$$

Bayes Theorem Proof

Claim:

$$P(A), P(B) > 0 \quad \Rightarrow \quad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

By definition of conditional probability

$$P(A \cap B) = P(A|B)P(B)$$

Swapping A, B gives

$$P(B \cap A) = P(B|A)P(A)$$

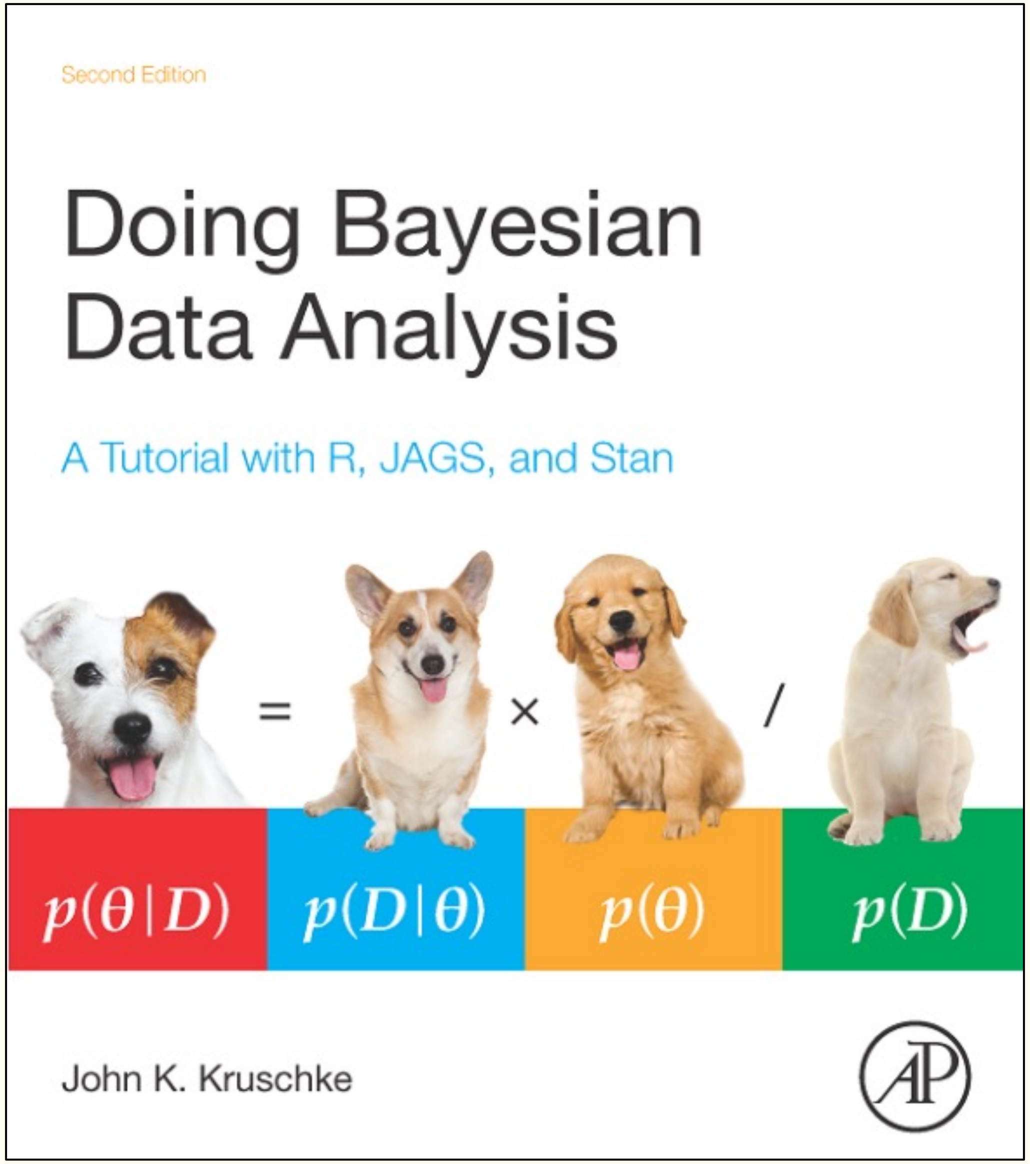
But $P(A \cap B) = P(B \cap A)$, so

$$P(A|B)P(B) = P(B|A)P(A)$$

Dividing both sides by $P(B)$ gives

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Brain Break


Second Edition
Doing Bayesian Data Analysis
A Tutorial with R, JAGS, and Stan
=
×
/
p(θ|D)
p(D|θ)
p(θ)
p(D)
John K. Kruschke

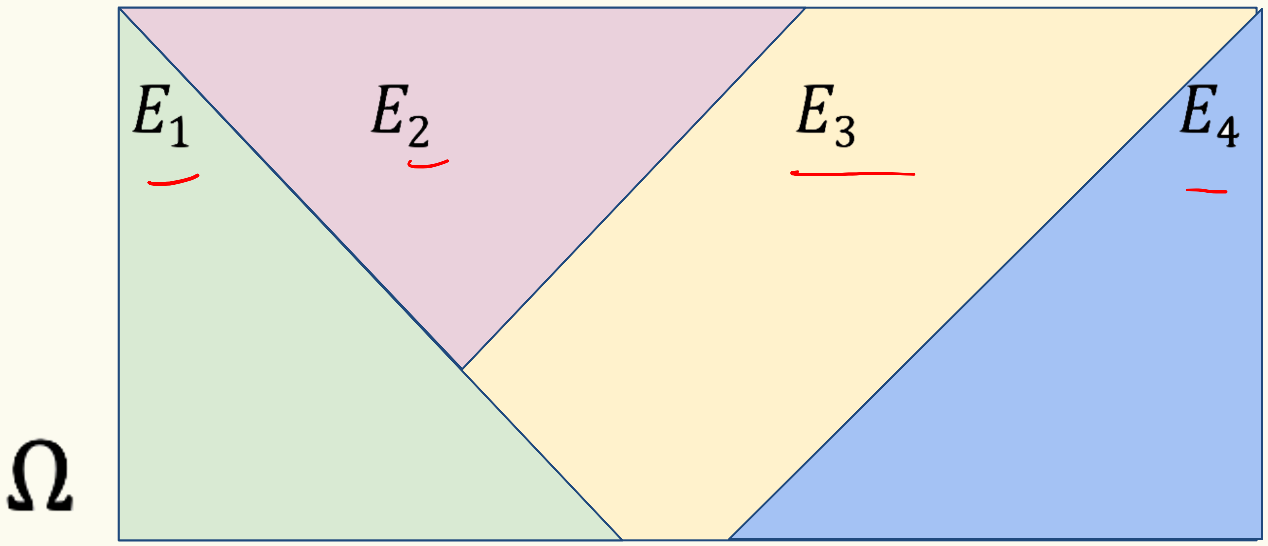
Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- More Examples

Partitions (Idea)

These events **partition** the sample space

1. They "cover" the whole space
2. They don't overlap



Partition

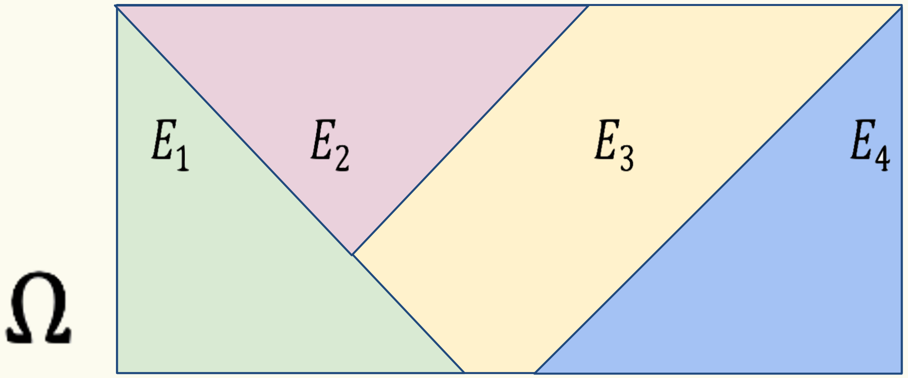
Definition. Non-empty events $E_1, E_2, \ldots, E_n$ **partition** the sample space Ω if

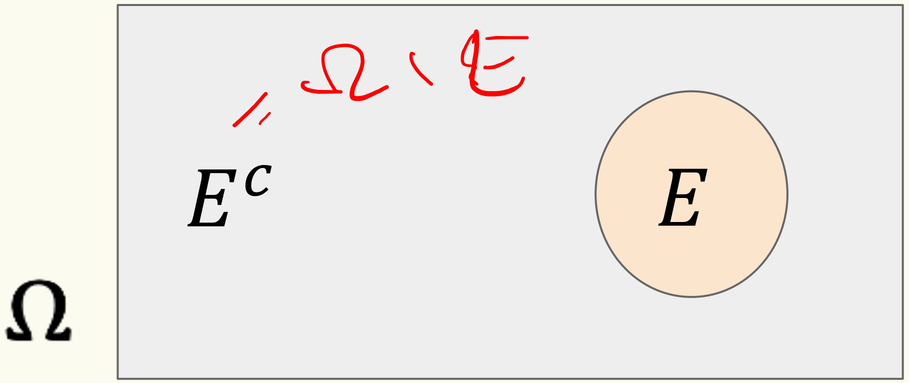
(Exhaustive)

$$E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$$

(Pairwise Mutually Exclusive)

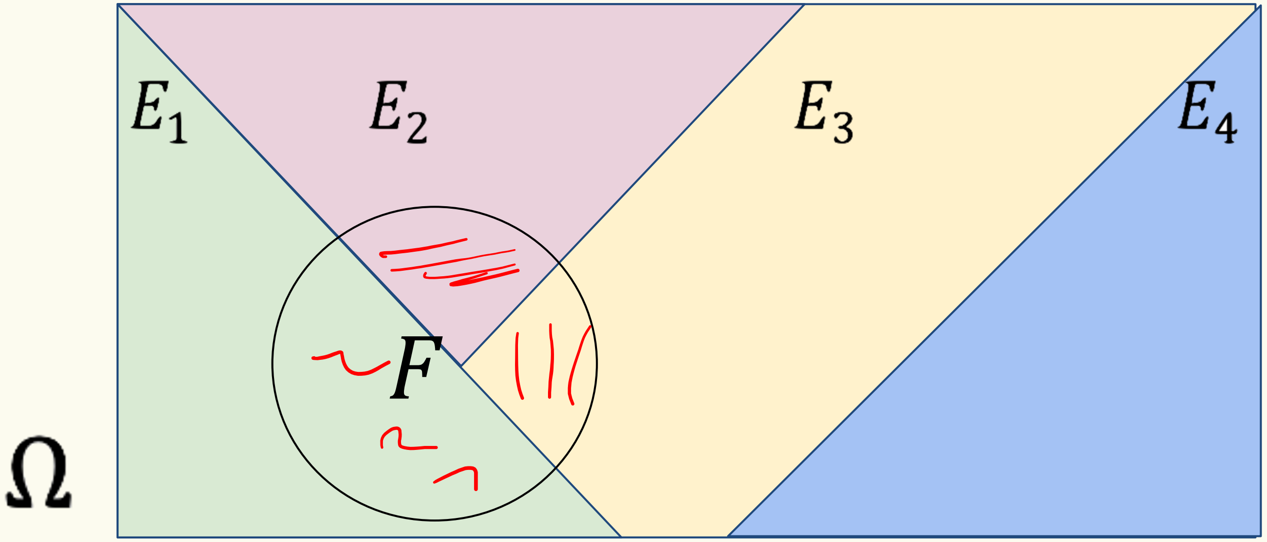
$$\forall_i \forall_{i \neq j} \; E_i \cap E_j = \emptyset$$





Law of Total Probability (Idea)

If we know $E_1, E_2, \ldots, E_n$ partition Ω, what can we say about $P(F)$?



$P(F)$

$P(F \cap E_1)$

$P(F \cap E_2)$

$P(F \cap E_3)$

$P(F \cap E_4) = 0$

Law of Total Probability (LTP)

Definition. If events $E_1, E_2, \ldots, E_n$ partition the sample space Ω, then for any event F

$$P(F) = P(F \cap E_1) + \ldots + P(F \cap E_n) = \sum_{i=1}^{n} P(F \cap E_i)$$

Using the definition of conditional probability $P(F \cap E) = P(F|E)P(E)$

We can get the alternate form of this that shows

$$P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

Another Contrived Example

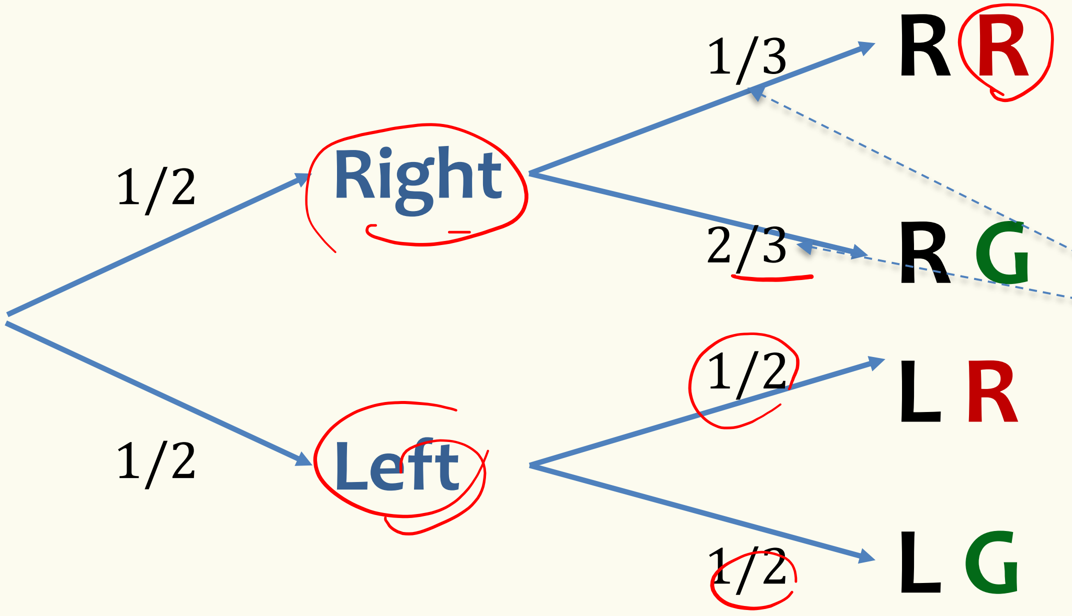
Alice has two pockets:

- **Left pocket:** Two red balls, two green balls
- **Right pocket:** One red ball, two green balls.

Alice picks a random ball from a random pocket.

[Both pockets equally likely, each ball equally likely.]

Sequential Process



- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

1/3 = $P(\mathbf{R}|\text{Right})$ and 2/3 = $P(\mathbf{G}|\text{Right})$

$$P(\mathbf{R}) = P(\mathbf{R} \cap \text{Left}) + P(\mathbf{R} \cap \text{Right})$$

(Law of total probability)

$$= P(\text{Left}) \times P(\mathbf{R}|\text{Left}) + P(\text{Right}) \times P(\mathbf{R}|\text{Right})$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- More Examples

Example – Zika Testing

Usually no or mild symptoms (rash); sometimes severe symptoms (paralysis).

During pregnancy: may cause birth defects.

Suppose you took a Zika test, and it returns "positive", what is the likelihood that you actually have the disease?

- Tests for diseases are rarely 100% accurate.

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $P(T|Z)$
- However, the test may yield a "false positive" 1% of the time $P(T|Z^c)$
- 0.5% of the US population has Zika. $P(Z)$

What is the probability you have Zika (event Z) if you test positive (event T).?

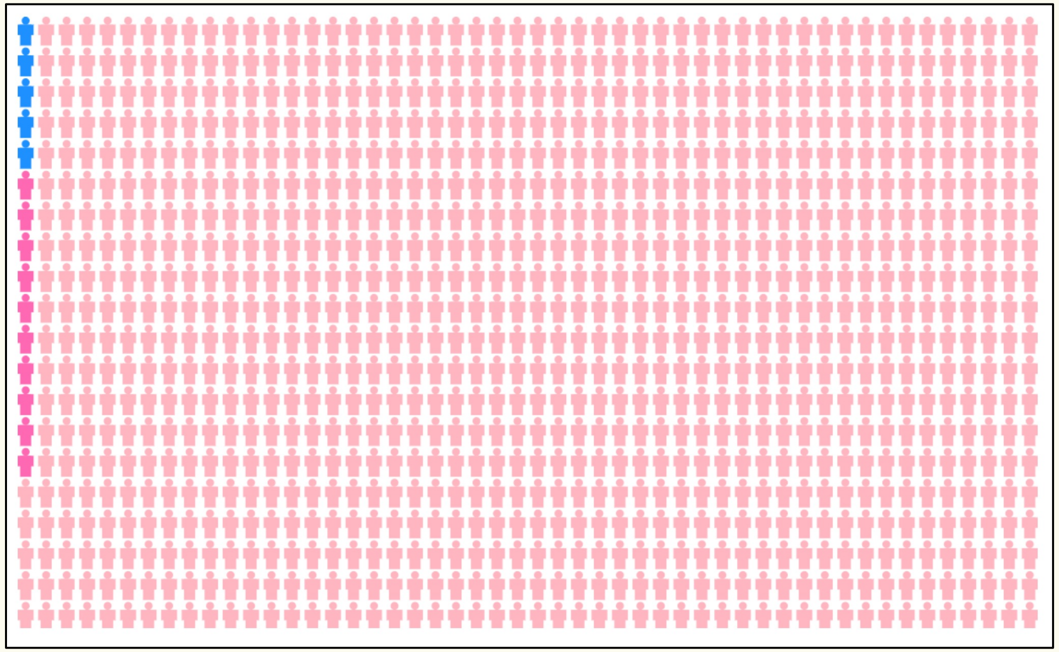
Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $P(T|Z)$
- However, the test may yield a "false positive" 1% of the time $P(T|Z^c)$
- 0.5% of the US population has Zika. $P(Z)$

500 have Zika
99,500 do not

What is the probability you have Zika (event Z) if you test positive (event T)?



Suppose we had 100,000 people:

- 490 **have Zika** and **test positive**
- 10 **have Zika** and **test negative**
- 98,505 **do not have Zika** and **test negative**
- 995 **do not have Zika** and **test positive**

$$\frac{490}{490 + 995} \approx 0.33$$

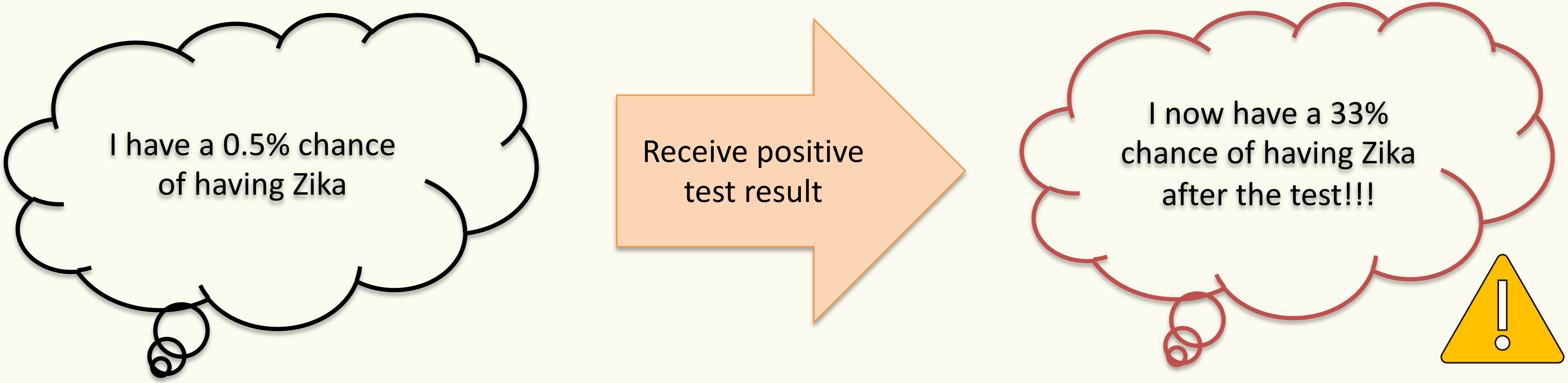
Demo

Philosophy – Updating Beliefs

While it's not 98% that you have the disease, your beliefs changed **drastically**

Z = you have Zika

T = you test positive for Zika



Prior: $P(Z)$

Posterior: $P(Z|T)$

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $P(T|Z)$
- However, the test may yield a "false positive" 1% of the time $P(T|Z^c)$
- 0.5% of the US population has Zika. $P(Z)$

What is the probability you test negative (event T^c) if you have Zika (event Z)?

$$P(T^c|Z) = 1 - P(T|Z) = 2\%$$

Conditional Probability Defines a Probability Space

The probability conditioned on $\mathcal{A}$ follows the same properties as (unconditional) probability.

Example. $P(\mathcal{B}^c|\mathcal{A}) = 1 - P(\mathcal{B}|\mathcal{A})$

Formally. (Ω, P) is a probability space and $P(\mathcal{A}) > 0$

⇒ $(\mathcal{A}, P(\cdot\,|\mathcal{A}))$ is a probability space