Lecture 4: Counting pigeons, counting practice
Last Class: Counting
• Binomial Coefficients
• Binomial Theorem
• Inclusion-Exclusion

Today:
• Pigeonhole Principle
• Counting practice
Inclusion-Exclusion

But what if the sets are not disjoint?

\[ |A| = 43 \]
\[ |B| = 20 \]
\[ |A \cap B| = 7 \]
\[ |A \cup B| = ? ? ? \]

**Fact.** \[ |A \cup B| = |A| + |B| - |A \cap B| \]
Inclusion-Exclusion Example: RSA

Last time: For (distinct) primes $p, q$, and $N = p \cdot q$, how many integers in $\{0, \ldots, N - 1\}$ have no common factor with $N$?

Idea:
- $A$ = integers $\{0, \ldots, N - 1\}$ divisible by $p$ = multiples of $p$ mod $N$
- $B$ = integers $\{0, \ldots, N - 1\}$ divisible by $q$ = multiples of $q$ mod $N$
- Wanted: $N - |A \cup B|$
Example: \( p = 3, q = 5 \quad N = 3 \times 5 \)

\[
B = \{0, 5, 10\}
\]

\[
A = \{0, 3, 6, 9, 12\}
\]

\[|B| = 3\]

\[|A| = 5\]

\[A \cap B\] contains multiples of 3 & 5 (mod 15) \[A \cap B = \{0\}\]

\# Integers between 0 and 14 that share a non-trivial divisor with 15
\[
= |A| + |B| - |A \cap B| = 3 + 5 - 1 = 7
\]

\# Integers between 0 and 14 that share no non-trivial divisor with 15
\[
= 15 - 7 = 8
\]
More general: Integers mod $N$ co-prime with $N = pq$ for $p, q$ prime

$B = \{0, q, 2q, \ldots, (p - 1)q\}$

$A = \{0, p, 2p, \ldots, (q - 1)p\}$

$|B| = p$

$|A| = q$

$A \cap B$ contains multiples of $p$ & $q \pmod{N}$  \quad \quad A \cap B = \{0\}$

# Integers between 0 and $N - 1$ that share a non-trivial divisor with $N$

$= |A| + |B| - |A \cap B| = p + q - 1$

# Integers between 0 and $N - 1$ that are co-prime with $N$

$= N - (p + q - 1) = pq - p - q + 1 = (p - 1)(q - 1)$
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Pigeonhole Principle (PHP): Idea

10 pigeons, 9 pigeonholes
If 11 children have to share 3 cakes, at least one cake must be shared by how many children?
Pigeonhole Principle – More generally

If there are $n$ pigeons in $k < n$ holes, then one hole must contain at least $\frac{n}{k}$ pigeons!

Proof. Assume there are $< \frac{n}{k}$ pigeons per hole.

Then, there are $< k \cdot \frac{n}{k} = n$ pigeons overall.

Contradiction!
Pigeonhole Principle – Better version

If there are \( n \) pigeons in \( k < n \) holes, then one hole must contain at least \( \left\lceil \frac{n}{k} \right\rceil \) pigeons!

**Reason.** Can’t have fractional number of pigeons

**Syntax reminder:**

- **Ceiling:** \([x]\) is \(x\) rounded up to the nearest integer (e.g., \([2.731] = 3\))
- **Floor:** \([x]\) is \(x\) rounded down to the nearest integer (e.g., \([2.731] = 2\))
Pigeonhole Principle: Strategy

To use the PHP to solve a problem, there are generally 4 steps:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP
Pigeonhole Principle – Example

In a room with 367 people, there are at least two with the same birthday.

Solution:

1. \(367\) pigeons = people
2. \(366\) holes (365 for a normal year + Feb 29) = possible birthdays
3. Person goes into hole corresponding to own birthday
4. By PHP, there must be two people with the same birthday
Pigeonhole Principle – Example (Surprising?)

In every set $S$ of 100 integers, there are at least two elements whose difference is a multiple of 37.

When solving a PHP problem:

1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP
Pigeonhole Principle – Example (Surprising?)

In every set $S$ of 100 integers, there are at least two elements whose difference is a multiple of 37.

When solving a PHP problem:
1. Identify pigeons
2. Identify pigeonholes
3. Specify how pigeons are assigned to pigeonholes
4. Apply PHP

Pigeons: integers $x$ in $S$
Pigeonholes: $\{0,1,...,36\}$

Assignment: $x$ goes to $x \mod 37$

Since $100 > 37$, by PHP, there are $x \neq y \in S$ s.t.

$x \mod 37 = y \mod 37$ which implies

$x - y = 37k$ for some integer $k$
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Quick Review of Cards

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

How many possible 5 card hands?

\[
\binom{52}{5}
\]
A "straight" is five consecutive rank cards of any suit (where A, 2, 3, 4, 5 also counts as consecutive). How many possible straights?

\[ 10 \cdot 4^5 = 10,240 \]
Counting Cards II

- A flush is five card hand all of the same suit.
  How many possible flushes?

\[ 4 \cdot \binom{13}{5} = 5148 \]
Counting Cards III

- A flush is five card hand all of the same suit. How many possible flushes?
  \[ 4 \cdot \binom{13}{5} = 5148 \]

- How many flushes are NOT straights?
  \[ = \#\text{flush} - \#\text{flush and straight} \]
  \[ (4 \cdot \binom{13}{5} = 5148) - 10 \cdot 4 \]
For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

\[
\binom{4}{3} \cdot \binom{49}{2}
\]

**Poll:**
A. **Correct**
B. Overcount
C. Undercount

https://pollev.com/stefanotessaro617
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First choose 3 Aces. Then choose remaining two cards.

\[
\binom{4}{3} \cdot \binom{49}{2}
\]

Problem: This counts a hand with all 4 Aces in 4 different ways! e.g. it counts $A\spadesuit, A\heartsuit, A\clubsuit, A\diamondsuit, 2\heartsuit$ four times:

\[
\begin{align*}
\{A\spadesuit, A\heartsuit, A\clubsuit\} & \{A\diamondsuit, 2\heartsuit\} \\
\{A\spadesuit, A\heartsuit, A\clubsuit\} & \{A\diamondsuit, 2\heartsuit\} \\
\{A\spadesuit, A\heartsuit, A\clubsuit\} & \{A\diamondsuit, 2\heartsuit\} \\
\{A\spadesuit, A\heartsuit, A\clubsuit\} & \{A\diamondsuit, 2\heartsuit\} \\
\end{align*}
\]
For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

**EXAMPLE:** How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

\[ \binom{4}{3} \cdot \binom{48}{2} + \binom{48}{1} \]

No sequence \(\Rightarrow\) under counting \hspace{1cm} Many sequences \(\Rightarrow\) over counting
8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column?

Sequential process:
1. Column for pawn
2. Row for pawn
3. Column for bishop
4. Row for bishop
5. Column for knight
6. Row for knight

$$(8 \cdot 7 \cdot 6)^2$$
Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don’t share a row or a column

\[
(8 \cdot 7)^2 / 2
\]

Fully ordered: Pretend Rooks are different
1. Column for rook1
2. Row for rook1
3. Column for rook2
4. Row for rook2

“Remove” the order of the two rooks:

\[
(8 \cdot 7)^2 / 2
\]
Counting when order only partly matters

We often want to count # of partly ordered lists:

Let $M =$ # of ways to produce fully ordered lists

$P =$ # of partly ordered lists

$N =$ # of ways to produce corresponding fully ordered list given a partly ordered list

Then $M = P \cdot N$ by the product rule. Often $M$ and $N$ are easy to compute:

$$P = \frac{M}{N}$$

Dividing by $N$ “removes” part of the order.
Anagrams (another look at rearranging SEATTLE)

How many ways can you arrange the letters in “Godoggy”?

\[ n = 7 \text{ Letters}, \ k = 4 \text{ Types \{G, O, D, Y\}} \]

\[ n_1 = 3, \ n_2 = 2, \ n_3 = 1, \ n_4 = 1 \]

\[
\frac{7!}{3!2!1!1!} = \binom{7}{3,2,1,1}
\]

Multinomial coefficients
Multinomial Coefficients

If we have $k$ types of objects ($n$ total), with $n_1$ of the first type, $n_2$ of the second, …, and $n_k$ of the $k^{\text{th}}$, then the number of orderings possible is

\[
\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}
\]
Counting using binary encoding*

The number of ways to distribute \( n \) indistinguishable balls into \( k \) distinguishable bins is

\[
\binom{n + k - 1}{k - 1} = \binom{n + k - 1}{n}
\]

E.g., = # of ways to add \( k \) non-negative integers up to \( n \)

*aka. “stars and bars method”*
How many ways can you distribute 32 identical coins among Alex, Barbara, Charlie, Dana, and Eve?

1. Identify balls
2. Identify bins

\[
\binom{32 + 5 - 1}{5 - 1}
\]
Binomial Theorem

**Theorem.** Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

**Corollary.**

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$
Binomial Theorem: A less obvious consequence

**Theorem.** Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

- $= -1$ if $k$ is odd
- $= +1$ if $k$ is even

**Corollary.** For every $n$, if $O$ and $E$ are the sets of odd and even integers between 0 and $n$

$$\sum_{k \in O} \binom{n}{k} = \sum_{k \in E} \binom{n}{k}$$

e.g., $n=4$: $1 4 6 4 1$

**Proof:** Set $x = -1, y = 1$ in the binomial theorem
Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Binary encoding/stars and bars