Lecture 3: More counting
Binomial Coefficients, Binomial Theorem, Inclusion-Exclusion
Announcements

• Office hours start today
  – In particular, I will be available for office hours starting right after class today (CSE 644)

• Problem Set 1
  – Read the first page for how to write up your homework solutions. Don’t wait until you are working on the questions to figure it out.
  – Section solutions are another good place to look at for examples.

• Resources
  – Textbook readings can provide another perspective
  – Office Hours
  – EdStem discussion

• EdStem discussion etiquette
  – OK to publically discuss content of the course and any confusion over topics discussed in class, but not solutions for current homework problems, or anything about current exams that have not yet been graded.
  – It is also acceptable to ask for clarifications about what current homework problems are asking and concepts behind them, just not about their solutions.
Recap of Last Time

**Permutations.** The number of orderings of $n$ distinct objects

$$n! = n \times (n - 1) \times \cdots \times 2 \times 1$$

Example: How many sequences in $\{1,2,3\}^3$ with no repeating elements?

**k-Permutations.** The number of orderings of **only** $k$ out of $n$ distinct objects

$$P(n,k) = n \times (n - 1) \times \cdots \times (n - k + 1) = \frac{n!}{(n-k)!}$$

Example: How many sequences of 5 distinct alphabet letters from $\{A, B, \ldots, Z\}$?

**Combinations / Binomial Coefficient.** The number of ways to select $k$ out of $n$ objects, where ordering of the selected $k$ does not matter:

$$\binom{n}{k} = \frac{P(n,k)}{k!} = \frac{n!}{k!(n-k)!}$$

Example: How many size-5 **subsets** of $\{A, B, \ldots, Z\}$?

Example: How many shortest paths from Gates to Starbucks?

Example: How many solutions $(x_1, \ldots, x_k)$ such that $x_1, \ldots, x_k \geq 0$ and $\sum_{i=1}^{k} x_i = n$?
Recap* Example – Counting Paths

A slightly modified example

Path $\in \{\uparrow, \rightarrow\}^7$

Example path: $(\uparrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \rightarrow)$

$\binom{7}{3} = \binom{7}{4}$
Agenda

- Binomial Coefficients
- Binomial Theorem
- Inclusion-Exclusion
Binomial Coefficient – Many interesting and useful properties

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}
\]

\[
\binom{n}{n} = 1 \\
\binom{n}{1} = n \\
\binom{n}{0} = 1
\]

**Fact.** \(\binom{n}{k} = \binom{n}{n-k}\) 

Symmetry in Binomial Coefficients

**Fact.** \(\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}\) 

Pascal’s Identity

**Fact.** \(\sum_{k=0}^{n} \binom{n}{k} = 2^n\) 

Follows from Binomial Theorem
Symmetry in Binomial Coefficients

Fact. \( \binom{n}{k} = \binom{n}{n-k} \)

This is called an Algebraic proof, i.e., Prove by checking algebra

Proof. \( \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k} \)

Why??
Symmetry in Binomial Coefficients – A different proof

Fact. \( \binom{n}{k} = \binom{n}{n-k} \)

Two equivalent ways to choose \( k \) out of \( n \) objects (unordered)
1. Choose which \( k \) elements are included
2. Choose which \( n - k \) elements are excluded

\[
\binom{4}{1} = 4 = \binom{4}{3}
\]
Symmetry in Binomial Coefficients – A different proof

Fact. \((\binom{n}{k}) = \binom{n}{n-k}\)

Two equivalent ways to choose \(k\) out of \(n\) objects (unordered)
1. Choose which \(k\) elements are included
2. Choose which \(n-k\) elements are excluded

Format for a combinatorial argument/proof of \(a = b\)
- Let \(S\) be a set of objects
- Show how to count \(|S|\) one way \(\Rightarrow |S| = a\)
- Show how to count \(|S|\) another way \(\Rightarrow |S| = b\)
Combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive
Pascal’s Identity

Fact. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

How to prove Pascal’s identity?

Algebraic argument:

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)! (n-k)!} + \frac{(n-1)!}{k! (n-1-k)!} \\
= \frac{(n-1)!}{(k-1)! (n-k)!} + \frac{(n-1)!}{k! (n-1-k)!} \\
= 20 \text{ years later} \ldots \\
= \frac{n!}{k! (n-k)!} \\
= \binom{n}{k}
\]

Hard work and not intuitive

Let’s see a combinatorial argument
Example – Pascal’s Identity

Fact. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

\[ |S| = |A| + |B| \]

Combinatorial proof idea:
• Find disjoint sets \( A \) and \( B \) such that \( A, B, \) and \( S = A \cup B \) have the sizes above.
• The equation then follows by the Sum Rule.
Example – Pascal’s Identity

Fact. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

\[ |S| = |A| + |B| \]

\( S \): set of size \( k \) subsets of \( \{1, 2, \ldots, n\} \).

e.g. \( n = 4, k = 2 \), \( S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\} \)

\( A \): set of size \( k \) subsets of \( \{n\} \) that DO include \( n \)

\( A = \{\{1,4\}, \{2,4\}, \{3,4\}\} \)

\( B \): set of size \( k \) subsets of \( \{n\} \) that DON’T include \( n \)

\( B = \{\{1,2\}, \{1,3\}, \{2,3\}\} \)

Combinatorial proof idea:

- Find disjoint sets \( A \) and \( B \) such that \( A, B, \) and \( S = A \cup B \) have these sizes.
Example – Pascal’s Identity

**Fact.** \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

\[ |S| = |A| + |B| \]

**Combinatorial proof idea:**
- Find **disjoint** sets \( A \) and \( B \) such that \( A, B, \text{ and } S = A \cup B \) have these sizes

- \( n \) is in set, need to choose other \( k - 1 \) elements from \([n - 1]\)
  \[ |A| = \binom{n - 1}{k - 1} \]

- \( n \) not in set, need to choose \( k \) elements from \([n - 1]\)
  \[ |B| = \binom{n - 1}{k} \]

**Sets:**
- \( S \): set of size \( k \) subsets of \([n] = \{1, 2, \ldots, n\} \).
- \( A \): set of size \( k \) subsets of \([n]\) that **DO** include \( n \)
- \( B \): set of size \( k \) subsets of \([n]\) that **DON’T** include \( n \)
Agenda

• Binomial Coefficients
• Binomial Theorem
• Inclusion-Exclusion
Binomial Theorem: Idea

\[(x + y)^2 = (x + y)(x + y)\]
\[= xx + xy + yx + yy\]
\[= x^2 + 2xy + y^2\]

\[(x + y)^4 = (x + y)(x + y)(x + y)(x + y)\]
\[= xyyyy + yyyyy + xyxy + yxyy + \ldots\]

\[x^4 + ? \cdot x^3y + ? \cdot x^2y^2 + \ldots\]

Poll: What is the coefficient for \(xy^3\)?
A. 4
B. \(\binom{4}{1}\)
C. \(\binom{4}{3}\)
D. 3

https://pollev.com/stefanotessaro617
Binomial Theorem: Idea

\[(x + y)^n = (x + y) \ldots (x + y)\]

Each term is of the form \(x^k y^{n-k}\), since each term is made by multiplying exactly \(n\) variables, either \(x\) or \(y\), one from each copy of \((x + y)\).

How many times do we get \(x^k y^{n-k}\)?

The number of ways to choose \(x\) from exactly \(k\) of the \(n\) copies of \((x + y)\) (the other \(n - k\) choices will be \(y\)) which is:

\[
{n \choose k} = {n \choose n - k}
\]
Binomial Theorem

**Theorem.** Let \( x, y \in \mathbb{R} \) and \( n \in \mathbb{N} \) a positive integer. Then,

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}
\]

Many properties of sums of binomial coefficients can be found by plugging in different values of \( x \) and \( y \) in the Binomial Theorem.

\[
x - y = 1
\]

**Corollary.**

\[
\sum_{k=0}^{n} \binom{n}{k} = 2^n
\]
Brain Break
Agenda

• Binomial Coefficients
• Binomial Theorem
• Inclusion-Exclusion
Recap Disjoint Sets

Sets that do not contain common elements ($A \cap B = \emptyset$)

Sum Rule: $|A \cup B| = |A| + |B|$
Inclusion-Exclusion

But what if the sets are not disjoint?

\[ |A| = 43 \]
\[ |B| = 20 \]
\[ |A \cap B| = 7 \]
\[ |A \cup B| = ??? \]

**Fact.** \[ |A \cup B| = |A| + |B| - |A \cap B| \]
Inclusion-Exclusion

What if there are three sets?

\[ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \]

Fact.

\[ |A| = 43 \quad |B| = 20 \quad |C| = 35 \quad |A \cap B| = 7 \quad |A \cap C| = 16 \quad |B \cap C| = 11 \quad |A \cap B \cap C| = 4 \quad |A \cup B \cup C| = ??? \]
Inclusion-Exclusion

Let $A, B$ be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \text{singles} - \text{doubles} + \text{triples} - \text{quads} + \cdots$$

$$= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|) + \cdots$$
Example: RSA

• In encrypting messages using RSA one starts with
  – Two big prime numbers $p$ and $q$ that are kept secret
  – Encodes messages using arithmetic $\mod N$ for $N = pq$.
  – One needs to work with numbers $\mod N$ that have no common factors with $N$ (“co-prime with $N$”)
    • Otherwise the secret leaks or decryption may not be defined uniquely.

  – To define RSA one needs to know how many such numbers there are...
Integers mod $N$ co-prime with $N = pq$ for $p, q$ prime

$B = \{0, q, 2q, \ldots, (p - 1)q\}$

$A = \{0, p, 2p, \ldots, (q - 1)p\}$

$|B| = p$

$|A| = q$

$A \cap B$ contains multiples of $p & q \pmod{N}$

$A \cap B = \{0\}$

# Integers between 0 and $N - 1$ that share a non-trivial divisor with $N$

$= |A| + |B| - |A \cap B| = p + q - 1 = |A \cup B|$

# Integers between 0 and $N - 1$ that are co-prime with $N$

$= \left\lceil \frac{N}{2} \right\rceil - (p + q - 1) = pq - p - q + 1 = (p - 1)(q - 1)$
Agenda

• Recap & Finish Binomial Coefficients
• Binomial Theorem
• Inclusion-Exclusion