CSE 312

Foundations of Computing II

Lecture 2: Permutation and Combinations

Announcements

Homework:

- Pset1 will be out before tomorrow's quiz section and is due 11:59pm next Wednesday.
- We will have the same pattern for all the other assignments except for the last one (because of the Memorial Day holiday).

Python programming on homework:

- Some problem sets will include coding problems
  - in Python (*no prior knowledge or experience required*)
  - provide a deeper understanding of how theory we discuss is used in practice
  - should be fun

Quick counting summary from last class

- Sum rule:

  If you can choose from

  - EITHER one of n options,
  - OR one of m options with NO overlap with the previous n,

  then the number of possible outcomes of the experiment is $n + m$

- Product rule:

  In a sequential process, if there are

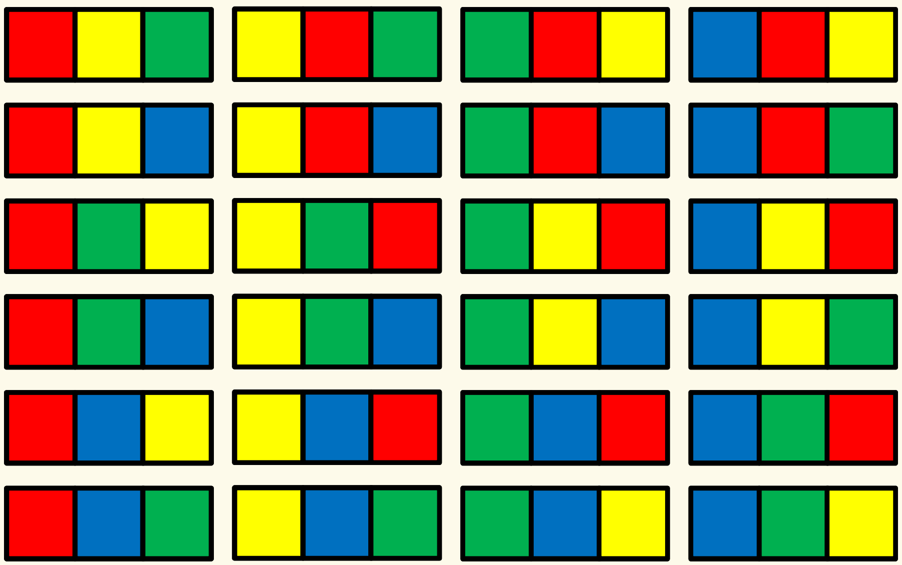
  - n_1 choices for the 1^{st} step,
  - n_2 choices for the 2^{nd} step (given the first choice), ..., and
  - n_k choices for the k^{th} step (given the previous choices),

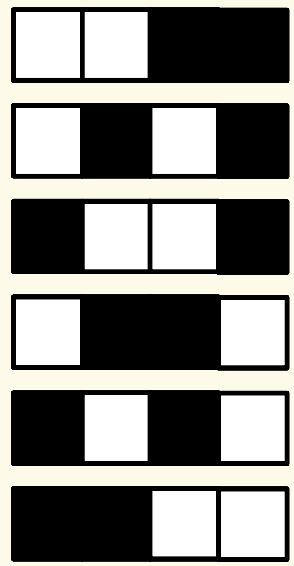
  then the total number of outcomes is $n_1 \times n_2 \times n_3 \times \cdots \times n_k$

- Representation of the problem is important (creative part)

Today: More Counting

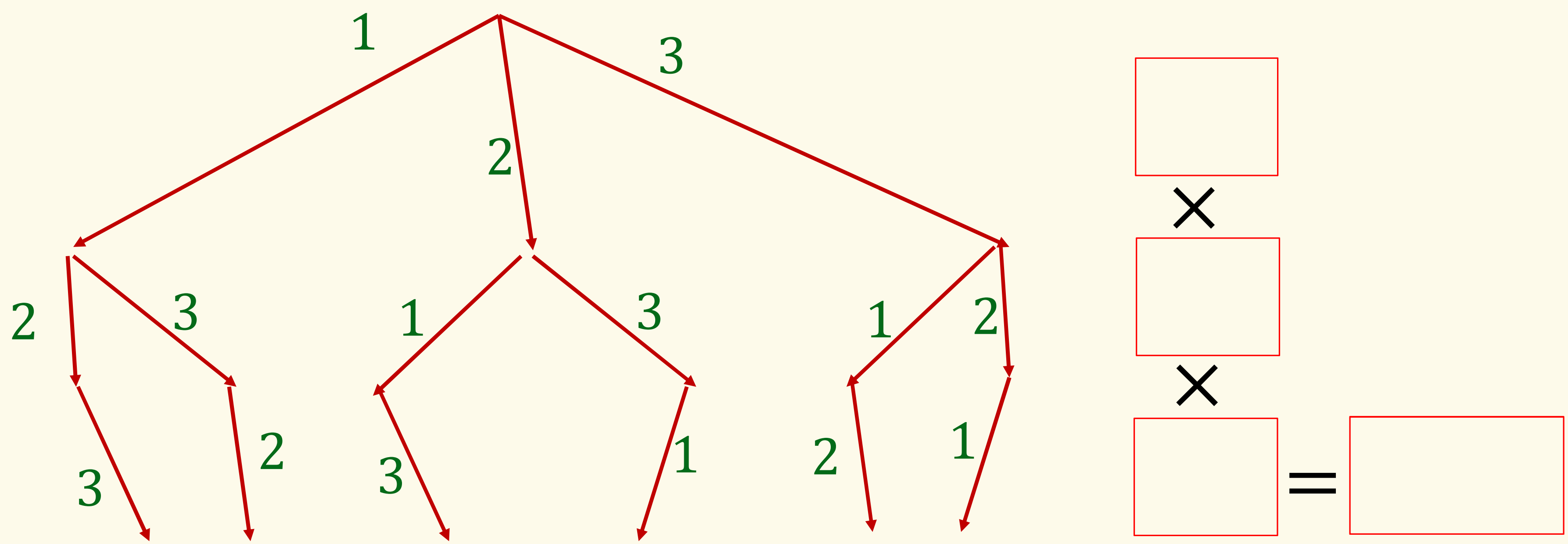
- Permutations and Combinations





Note: Sequential process for product rule works even if the sets of options are different at each point

"How many sequences in $\{1,2,3\}^3$ *with no repeating elements?"*



Nice use of sum rule: Counting using complements

"How many sequences in $\{1,2,3\}^3$ *have repeating elements?"* m

"# of sequences in $\{1,2,3\}^3$ *with no repeating elements"* $n =$ ☐

"# of sequences in $\{1,2,3\}^3$ $3^3 = 27$ $= m + n$ *by the sum rule*

All sequences

$m = 27 - n =$ ☐

Factorial

"How many ways to order elements in S, where $|S| = n$?" **Permutations**

Answer = $n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$

Definition. The **factorial function** is

$$n! = n \times (n-1) \times \cdots \times 2 \times 1$$

Note: $0! = 1$

Theorem. (Stirling's approximation)

$$\underbrace{\sqrt{2\pi}}_{= 2.5066} \cdot n^{n+\frac{1}{2}} \cdot e^{-n} \leq n! \leq \underbrace{e}_{= 2.7183} \cdot n^{n+\frac{1}{2}} \cdot e^{-n}$$

Huge: Grows exponentially in n

Distinct Letters

"How many sequences of 5 *distinct alphabet letters from* $\{A, B, \ldots, Z\}$*?"*

E.g., AZURE, BINGO, TANGO. But not: STEVE, SARAH

Answer: $26 \times 25 \times 24 \times 23 \times 22 = 7893600$

In general

Aka: k-permutations

Fact. # of k-element sequences of distinct symbols from an n-element set is

$$P(n, k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Number of Subsets

"How many size-5 ***subsets*** *of* $\{A, B, \ldots, Z\}$*?"*

E.g., {A,Z,U,R,E}, {B,I,N,G,O}, {T,A,N,G,O}. But not: {S,T,E,V}, {S,A,R,H},…

Difference from k-permutations: NO ORDER

Different sequences: TANGO, OGNAT, ATNGO, NATGO, ONATG …

Same set: {T,A,N,G,O}, {O,G,N,A,T}, {A,T,N,G,O}, {N,A,T,G,O}, {O,N,A,T,G}… …

Number of Subsets – Idea

Consider a sequential process:

1. Choose a subset $S \subseteq \{A, B, \dots, Z\}$ of size $|S| = 5$
   e.g. $S = \{A, G, N, O, T\}$
2. Choose a permutation of letters in S
   e.g., *TANGO, AGNOT, NAGOT, GOTAN, GOATN, NGOAT, …*

Outcome: A sequence of 5 distinct letters from $\{A, B, \dots, Z\}$

$$??? \times 5! = \frac{26!}{21!}$$

$$??? = \frac{26!}{21!\,5!} = 65780$$

Number of Subsets – Binomial Coefficient

Fact. The number of subsets of size k of a set of size n is

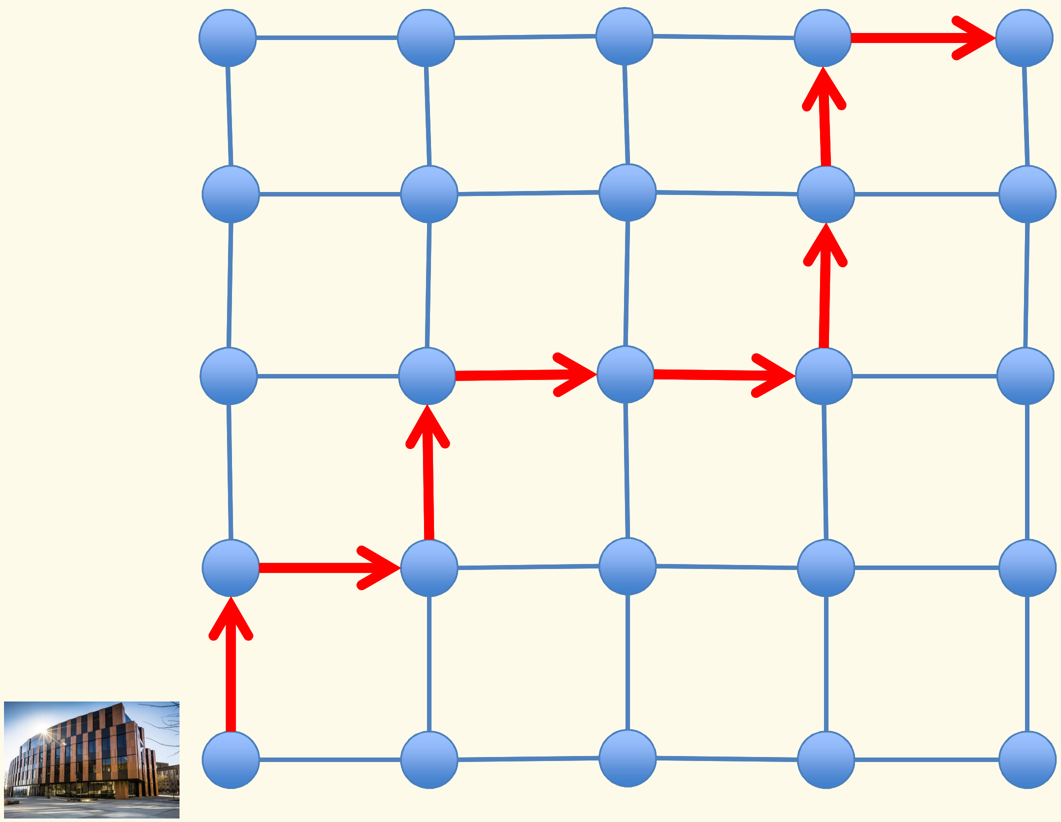
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial coefficient (verbalized as "n choose k")

Notation: $\binom{S}{k}$ = set of all k-element subsets of S. $\left|\binom{S}{k}\right| = \binom{|S|}{k}$

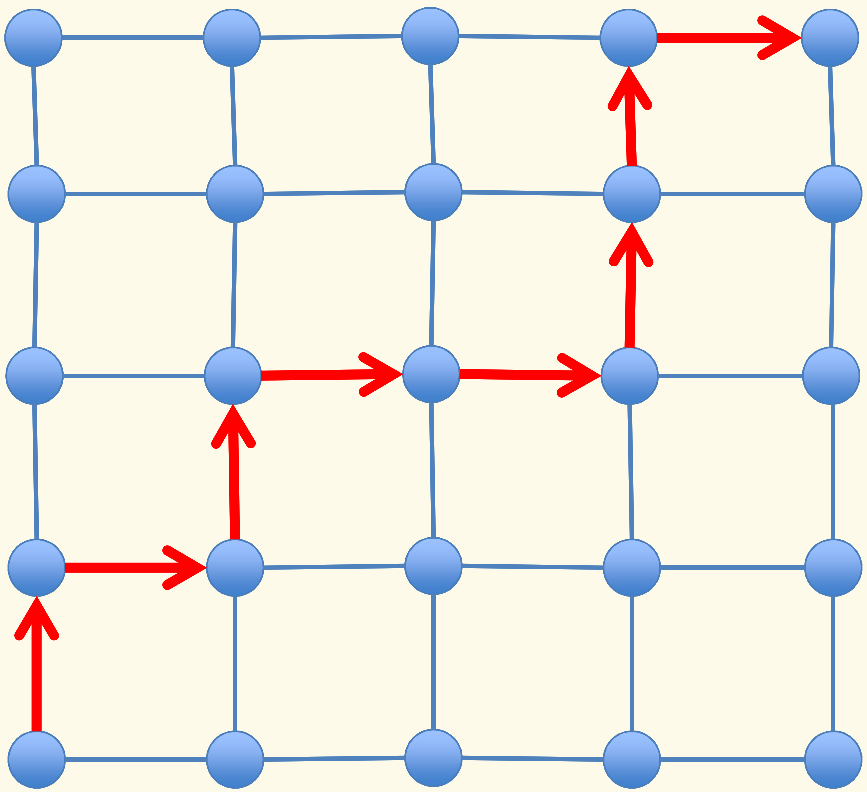
[also called **combinations**]

Example – Counting Paths



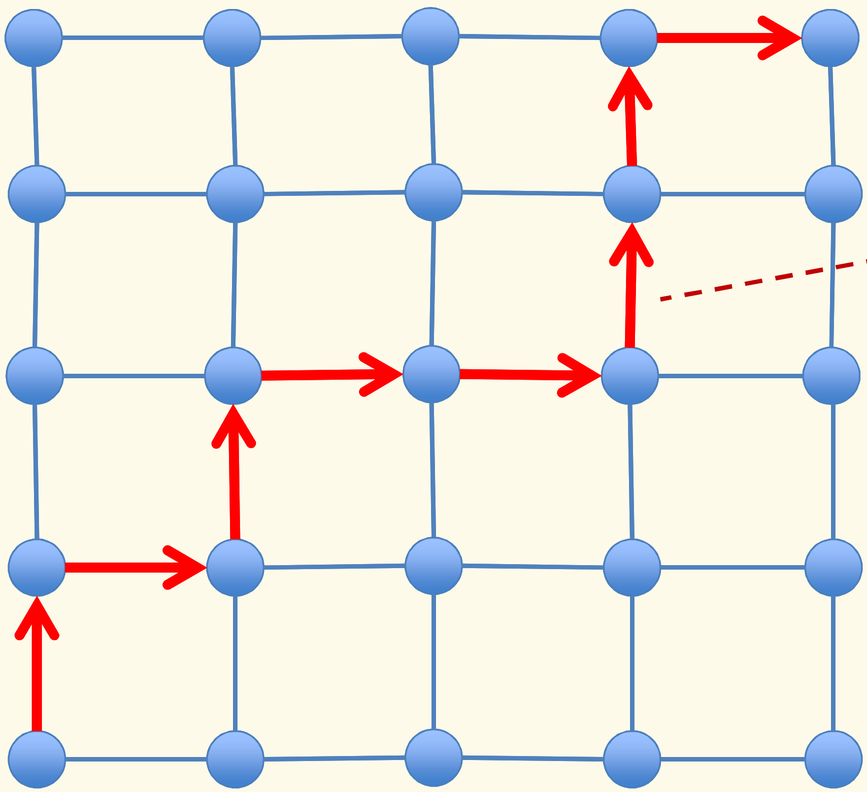
"How many shortest paths from Gates to Starbucks?"

Example – Counting Paths



How do we represent a shortest path?

Example – Counting Paths



$\text{Path} \in \{\uparrow, \rightarrow\}^8$

$(\uparrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \uparrow, \rightarrow)$

↑'s =4, # →'s=4

Poll:

A. 2^8

B. $\frac{8!}{4!}$

C. $\binom{8}{4} = \frac{8!}{4!4!}$

D. *No idea*

https://pollev.com/paulbeame028

Example – Sum of integers

"How many solutions $(x_1, \dots, x_k)$ *such that* $x_1, \dots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

Example: $k = 3$, $n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \dots$

Hint: we can represent each solution as a binary string.

Example – Sum of integers

Example: $k = 3$, $n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots$

Clever representation of solutions

$(3,1,1)$ → $1\ 1\ 1\ 0\ 1\ 0\ 1$

$(2,1,2)$ → $1\ 1\ 0\ 1\ 0\ 1\ 1$

$(1,0,4)$ → $1\ 0\ 0\ 1\ 1\ 1\ 1$

Example – Sum of integers

Example: $k = 3$, $n = 5$

sols = # strings from $\{0,1\}^7$ w/ exactly two 0s $= \binom{7}{2} = 21$

Clever representation of solutions

$(3,1,1)$ → $1\,1\,1\,0\,1\,0\,1$

$(2,1,2)$ → $1\,1\,0\,1\,0\,1\,1$

$(1,0,4)$ → $1\,0\,0\,1\,1\,1\,1$

Example – Sum of integers

"How many solutions $(x_1, \dots, x_k)$ *such that* $x_1, \dots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

sols = # strings from $\{0,1\}^{n+k-1}$ w/ $k-1$ 0s

$$= \binom{n+k-1}{k-1}$$

After a change in representation, the problem magically reduces to counting combinations.

Example – Word Permutations

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

Guess: 7! Correct?!

No! e.g., swapping two T's also leads to *SEATTLE*
swapping two E's also leads to *SEATTLE*

Counted as separate permutations, but they lead to the same word.

Example – Word Permutations

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

$$\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1$$

Locations of T's

Locations of E's

Location of S

Location of A

Location of L

T E S T A L E

Example II – Word Permutations

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

$$\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1 = \frac{7!}{2!\,\cancel{5!}} \times \frac{\cancel{5!}}{2!\,\cancel{3!}} \times \cancel{3!}$$

$$= \frac{7!}{2!\,2!} = 1260$$

Another interpretation:

Arrange the 7 letters as if they were distinct. Then divide by 2! to account for 2 duplicate T's, and divide by 2! again for 2 duplicate E's.

Quick Summary

- k-sequences: How many length k sequences over alphabet of size n?
  - Product rule ➔ n^k
- k-permutations: How many length k sequences over alphabet of size n, without repetition?
  - Permutation ➔ $\frac{n!}{(n-k)!}$
- k-combinations: How many size k subsets of a set of size n (without repetition and without order)?
  - Combination ➔ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

$$\binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

Pascal's Identity

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

Follows from Binomial theorem
(Next lecture)

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

This is called an Algebraic proof,
i.e., Prove by checking algebra

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

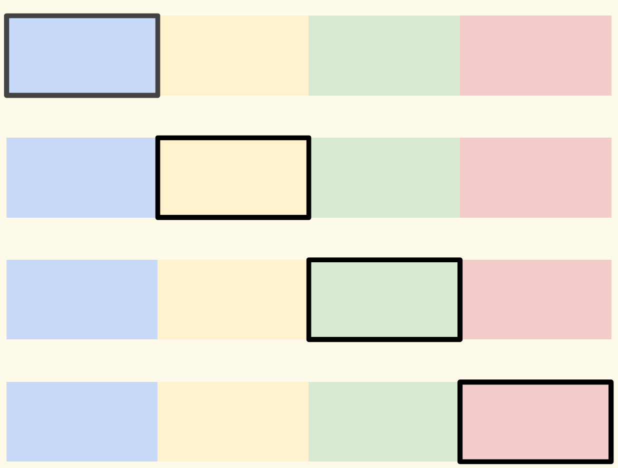
Why??

Symmetry in Binomial Coefficients – A different proof

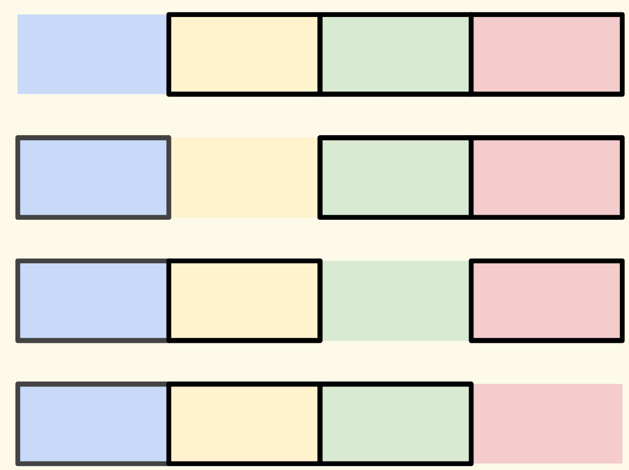
Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n - k$ elements are excluded



$$\binom{4}{1} = 4 = \binom{4}{3}$$



Symmetry in Binomial Coefficients – A different proof

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n - k$ elements are excluded

Format for a combinatorial argument/proof of $a = b$

- Let S be a set of objects
- Show how to count $|S|$ one way $\Rightarrow |S| = a$
- Show how to count $|S|$ another way $\Rightarrow |S| = b$

Combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive

Pascal's Identities

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!}$$
$$= 20\ years\ later\ \ldots$$
$$= \frac{n!}{k!\,(n-k)!}$$
$$= \binom{n}{k}$$

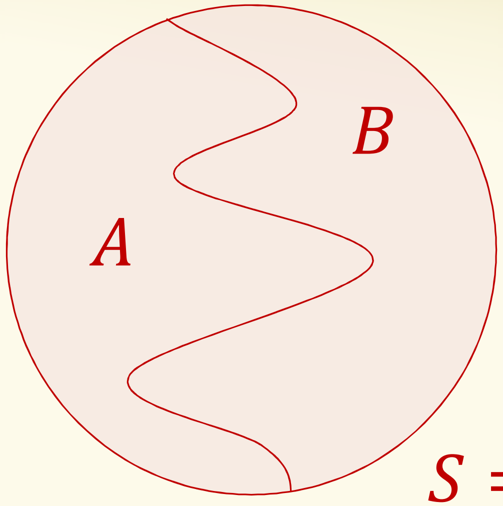
Hard work and not intuitive

Let's see a combinatorial argument

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| = |A| + |B|$$



$S = A \cup B$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have the sizes above.
- The equation then follows by the Sum Rule.

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$|S| \quad = \quad |A| \quad + \quad |B|$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

$|S| = \binom{n}{k}$

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

e.g. $n = 4, k = 2$, $S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: set of size k subsets of $[n]$ that DO include n

$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}$

B: set of size k subsets of $[n]$ that DON'T include n

$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| = |A| + |B|$$

Combinatorial proof idea:

- Find *disjoint* sets A and B such that A, B, and $S = A \cup B$ have these sizes

S: set of size k subsets of $[n] = \{1, 2, \cdots, n\}$.

n is in set, need to choose other $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

A: set of size k subsets of $[n]$ that DO include n

n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

B: set of size k subsets of $[n]$ that DON'T include n