

## Quiz Section 4

### Review

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1) **Probability Mass.** For every random variable  $X$ , we have  $\sum_x \mathbb{P}[X = x] = \underline{\hspace{2cm}}$ .

2) **Expectation.**  $\mathbb{E}[X] = \underline{\hspace{2cm}}$ .

3) **Linearity of expectation.** For any random variables  $X_1, \dots, X_n$ , and real numbers  $a_1, \dots, a_n$ ,

$$\mathbb{E}[a_1X_1 + \dots + a_nX_n] = \underline{\hspace{2cm}} .$$

4) **Variance.**  $\text{Var}(X) = \underline{\hspace{2cm}}$ .

5) **Independence.** Two random variables  $X$  and  $Y$  are **independent** if  $\underline{\hspace{2cm}}$ .

6) **Variance and Independence.** For any two independent random variables  $X$  and  $Y$ ,  $\text{Var}(X + Y) = \underline{\hspace{2cm}}$ .

### Task 1 – Identify that range!

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Identify the support/range  $\Omega_X$  of the random variable  $X$ , if  $X$  is...

- a) The sum of two rolls of a six-sided die.
- b) The number of lottery tickets I buy until I win it.
- c) The number of heads in  $n$  flips of a coin with  $0 < \mathbb{P}(\text{head}) < 1$ .
- d) The number of heads in  $n$  flips of a coin with  $\mathbb{P}(\text{head}) = 1$ .

### Task 2 – Symmetric Difference

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Suppose  $A$  and  $B$  are random, independent (possibly empty) subsets of  $\{1, 2, \dots, n\}$ , where each subset is equally likely to be chosen as  $A$  or  $B$ . Consider  $A\Delta B = (A \cap B^C) \cup (B \cap A^C) = (A \cup B) \cap (A^C \cup B^C)$ , i.e., the set containing elements that are in exactly one of  $A$  and  $B$ . Let  $X$  be the random variable that is the size of  $A\Delta B$ . What is  $\mathbb{E}[X]$ ?

### Task 3 – Hungry Washing Machine

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You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let  $X$  be the number of complete pairs of socks that you have left.

- a) What is the range of  $X$ ,  $\Omega_X$  (the set of possible values it can take on)? What is the probability mass function of  $X$ ?
- b) Find  $\mathbb{E}[X]$  from the definition of expectation.
- c) Find  $\mathbb{E}[X]$  using linearity of expectation.
- d) Which way was easier? Doing both (a) and (b), or just (c)?

## Task 4 – Balls in Bins

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Let  $X$  be the number of bins that remain empty when  $m$  balls are distributed into  $n$  bins randomly and independently. For each ball, each bin has an equal probability of being chosen. (Notice that two bins being empty are not independent events: if one bin is empty, that decreases the probability that the second bin will also be empty. This is particularly obvious when  $n = 2$  and  $m > 0$ .) Find  $\mathbb{E}[X]$ .

## Task 5 – Frogger

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A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability  $p_1$ , to the left with probability  $p_2$ , and doesn't move with probability  $p_3$ , where  $p_1 + p_2 + p_3 = 1$ . After 2 seconds, let  $X$  be the location of the frog.

- Find  $p_X(k)$ , the probability mass function for  $X$ .
- Compute  $\mathbb{E}[X]$  from the definition.
- Compute  $\mathbb{E}[X]$  again, but using linearity of expectation.

## Task 6 – 3-sided Die

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Let the random variable  $X$  be the sum of two independent rolls of a fair 3-sided die. (If you are having trouble imagining what that looks like, you can use a 6-sided die and change the numbers on 3 of its faces.)

- What is the probability mass function of  $X$ ?
- Find  $\mathbb{E}[X]$  directly from the definition of expectation.
- Find  $\mathbb{E}[X]$  again, but this time using linearity of expectation.
- What is  $\text{Var}(X)$ ?

## Task 7 – Practice

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- Let  $X$  be a random variable with  $p_X(k) = ck$  for  $k \in \{1, \dots, 5\} = \Omega_X$ , and 0 otherwise. Find the value of  $c$  that makes  $X$  follow a valid probability distribution and compute its mean and variance ( $\mathbb{E}[X]$  and  $\text{Var}(X)$ ).
- Let  $X$  be *any* random variable with mean  $\mathbb{E}[X] = \mu$  and variance  $\text{Var}(X) = \sigma^2$ . Find the mean and variance of  $Z = \frac{X - \mu}{\sigma}$ . (When you're done, you'll see why we call this a "standardized" version of  $X$ !)
- Let  $X, Y$  be independent random variables. Find the mean and variance of  $X - 3Y - 5$  in terms of  $\mathbb{E}[X], \mathbb{E}[Y], \text{Var}(X)$ , and  $\text{Var}(Y)$ .
- Let  $X_1, \dots, X_n$  be independent and identically distributed (iid) random variables each with mean  $\mu$  and variance  $\sigma^2$ . The sample mean is  $\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ . Find the mean and variance of  $\bar{X}$ . If you use the independence assumption anywhere, **explicitly label** at which step(s) it is necessary for your equalities to be true.

## Task 8 – Expectations, Independence, and Variance

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- Let  $U$  be a random variable which is uniform over the set  $[n] = \{1, 2, \dots, n\}$ , i.e,  $\mathbb{P}[U = i] = \frac{1}{n}$  for all  $i \in [n]$ . Compute  $\mathbb{E}[U^2]$  and  $\text{Var}(U)$ .  
**Hint:**  $\sum_{i=1}^n i = \frac{n(n+1)}{2}$  and  $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ .
- Let  $Y_1$  and  $Y_2$  be the independent outcomes of two dice rolls, and let  $Z = Y_1 + Y_2$ . Then, compute  $\mathbb{E}[Z^2]$  and  $\text{Var}(Z)$ .  
**Hint:** Try to use an indirect solution using linearity and independence, without the need of explicitly giving the distribution of  $Z^2$ .