Quiz Section 2 – Solutions

Review

1) Binomial theorem. $\forall x, y \in \mathbb{R}, \forall n \in \mathbb{N}$: $(x+y)^n =$ ______________________.

2) Inclusion-exclusion. $|A \cup B| =$ ______________________.

3) Inclusion-exclusion. $|A \cup B \cup C| =$ ____________________________________.

4) Pigeonhole principle. If there are n pigeons and k holes, and $n > k$, some hole has at least ______________________ pigeons.

5) Multinomial coefficients. Suppose there are n objects, but only k are distinct, with $k \leq n$. (For example, "godoggy" has $n = 7$ objects (characters) but only $k = 4$ are distinct: (g, o, d, y)). Let n_i be the number of times object i appears, for $i \in \{1, 2, \ldots, k\}$. (For example, $(3, 2, 1, 1)$, continuing the "godoggy" example.) The number of distinct ways to arrange the n objects is: ____________________________________

6) Binary encoding. The number of ways to distribute n indistinguishable balls into k distinguishable bins is ____________________________________.

7) Probability space. In a probability space $(\Omega, \mathbb{P})$, we have $\mathbb{P}(\omega)$ _______ for all $\omega \in \Omega$ and $\sum_{\omega \in \Omega} \mathbb{P}[\omega] =$ _______.

8) Mutually exclusive events. The events $\mathcal{A}$ and $\mathcal{B}$ are *mutually exclusive* if $\mathcal{A} \cap \mathcal{B} =$ ___________

9) Additivity of Probability. If $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are mutually exclusive events, then

$$\mathbb{P}\left[\bigcup_{i=1}^{n} \mathcal{A}_i\right] = \underline{\hspace{4cm}} .$$

10) Complement. For any event $\mathcal{A}$, $\mathbb{P}[\mathcal{A}^c] =$ _________________ .

11) Equally Likely Outcomes. If every outcome in a finite sample space Ω is equally likely, and E is an event, then $\mathbb{P}(E) =$ _________________ .

Task 1 – Binomial Theorem

What is the coefficient of z^{36} in $(-2x^2yz^3 + 5uv)^{312}$?

By the Binomial Theorem,

$$(-2x^2yz^3 + 5uv)^{312} = \sum_{k=0}^{312} \binom{312}{k} (-2x^2yz^3)^k (5uv)^{312-k} = \sum_{k=0}^{312} \binom{312}{k} (-2)^k x^{2k} y^k z^{3k} (5uv)^{312-k}$$

The term that gives z^{36} is the one with $k = 12$. Therefore, the coefficient is $\boxed{\binom{312}{12}(-2x^2y)^{12}(5uv)^{300}.}$

Task 2 – Ingredients

Find the number of ways to rearrange the word "INGREDIENT", such that no two identical letters are adjacent to each other. For example, "INGREEDINT" is invalid because the two E's are adjacent.

We use inclusion-exclusion. Let Ω be the set of all anagrams (permutations) of "INGREDIENT", and A_I be the set of all anagrams with two consecutive I's. Define A_E and A_N similarly. $A_I \cup A_E \cup A_N$ clearly are the set of anagrams we don't want. So we use complementing to count the size of $\Omega \backslash (A_I \cup A_E \cup A_N)$. By inclusion exclusion, $|A_I \cup A_E \cup A_N|$ =singles-doubles+triples, and by complementing, $|\Omega \backslash (A_I \cup A_E \cup A_N)| = |\Omega| - |A_I \cup A_E \cup A_N|$.

First, $|\Omega| = \frac{10!}{2!2!2!}$ because there are 2 of each of I,E,N's (multinomial coefficient). Clearly, the size of A_I is the same as A_E and A_N. So $|A_I| = \frac{9!}{2!2!}$ because we treat the two adjacent I's as one entity. We also need $|A_I \cap A_E| = \frac{8!}{2!}$ because we treat the two adjacent I's as one entity and the two adjacent E's as one entity (same for all doubles). Finally, $|A_I \cap A_E \cap A_N| = 7!$ since we treat each pair of adjacent I's, E's, and N's as one entity.

Putting this together gives $\boxed{\frac{10!}{2!2!2!} - \left(\binom{3}{1} \cdot \frac{9!}{2!2!} - \binom{3}{2} \cdot \frac{8!}{2!} + \binom{3}{3} \cdot 7!\right)}$

Task 3 – The Pigeonhole Principle

Show that in any group of n people there are two who have an identical number of friends within the group. (Friendship is bi-directional – i.e., if A is friend of B, then B is friend of A – and nobody is a friend of themselves.)

Solve in particular the following two cases individually:

a) Everyone has at least one friend.

Everyone has between 1 and $n-1$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, two of them will have the same number of friends.

b) At least one person has no friends.

Here, we need to observe that if someone has 0 friends, then nobody has $n-1$ friends (by the symmetry of the friendship relation). Then, possible choices are now between 0 and $n-2$ friends (i.e., $n-1$ holes), and there are n people (the "pigeons"). Therefore, two of them will have the same number of friends.

Task 4 – Card Party

At a card party, someone brings out a deck of bridge cards (4 suits with 13 cards in each). N people each pick 2 cards from the deck and hold onto them. What is the minimum value of N that guarantees at least 2 people have the same combination of suits?

$\boxed{N = 11}$: There are $\binom{4}{2}$ combinations of 2 different suits, plus 4 possibilities of having 2 cards of the same suit. With $N = 11$ you can apply the pigeonhole principle.

Task 5 – Balls from an Urn

Say an urn (a fancy name for a jar that doesn't have a lid) contains one red ball, one blue ball, and one green ball. (Other than for their colors, balls are identical.) Imagine we draw two balls *with replacement*, i.e., after drawing one ball, with put it back into the urn, before we draw the second one. (In particular, each ball is equally likely to be drawn.)

a) Give a probability space describing the experiment.

$\Omega = \{B, R, G\}^2$ and $\mathbb{P}[\omega] = 1/9$ for all $\omega \in \Omega$.

b) What is the probability that both balls are red? (Describe the event first, before you compute its probability.)

The event is $\mathcal{A} = \{RR\}$. Its probability is $\mathbb{P}[\mathcal{A}] = \frac{|\mathcal{A}|}{9} = \frac{1}{9}$.

c) What is the probability that at most one ball is red?

This is just $\mathcal{A}^c$, the complement of $\mathcal{A}$. We know that $\mathbb{P}[\mathcal{A}^c] = 1 - \mathbb{P}[\mathcal{A}] = 1 - \frac{1}{9} = 8/9$.

d) What is the probability that we get at least one green ball?

This is the event $\mathcal{B} = \{GR, GB, GG, RG, BG\}$, and thus $\mathbb{P}[\mathcal{B}] = \frac{|\mathcal{B}|}{9} = \frac{5}{9}$.

e) Repeat **c)-d)** for the case where the balls are drawn *without replacement*, i.e., when the first ball is drawn, it is not placed back from the urn.

Here, the probability space changes: First of all, the outcomes RR, GG, BB are not possible any more, so let us remove them from Ω, which is now $\Omega = \{BG, BR, GB, GR, RB, RG\}$. Note that now we have $\mathbb{P}[\omega] = 1/3 \cdot 1/2 = 1/6$ for every outcome, because we have three choices for the first ball, but only two for the second.

It can never be that both balls are red – therefore, for **b)**, the probability becomes 1 (i.e., the associated event *is* Ω.) For **c)**, instead, the event becomes $\mathcal{B} = \{GR, GB, RG, BG\}$, and $\mathbb{P}[\mathcal{B}] = 4 \cdot \frac{1}{6} = \frac{2}{3}$.

Task 6 – Congressional Tea

Twenty politicians are having tea, 6 Democrats and 14 Republicans.

a) If they only give tea to 10 of the 20 people, what is the probability that they only give tea to Republicans? (We assume every possible way of giving tea is equally likely.)

The sample space is the number of ways to give tea to people, so there are $\binom{20}{10}$ ways. The event is the ways to give tea to only Republicans, of which there are $\binom{14}{10}$ ways. So the probability is $\frac{\binom{14}{10}}{\binom{20}{10}}$.

b) If they only give tea to 10 of the 20 people, what is the probability that they give tea to 8 Republicans and 2 Democrats? (We assume every possible way of giving tea is equally likely.)

Similarly to the previous part, $\frac{\binom{14}{8}\binom{6}{2}}{\binom{20}{10}}$.

Task 7 – Shuffling Cards

We have a deck of cards, with 4 suits, and 13 cards in each suit. Within each suit, the cards are ordered Ace $>$ King $>$ Queen $>$ Jack $> 10 > \cdots > 2$. Also, suppose we perfectly shuffle the deck (i.e., all possible shuffles are equally likely).

What is the probability the first card on the deck is (strictly) larger than the second one?

First off, the sample space Ω here consists of all pairs of cards – which we can represent by their value *and* suit, e.g., $(4\clubsuit, A\diamondsuit)$. There $52 \cdot 51 = 2652$ possible outcomes, therefore $\mathbb{P}[\omega] = \frac{1}{2652}$ for all $\omega \in \Omega$.

Let us now look at the size of the event $\mathcal{E}$ containing all pairs where the first card is strictly larger than the second. Then, the number of pairs of values of cards a and b where $a < b$ is exactly $\binom{13}{2} = 13 \cdot 6 = 78$. We can then assign suits to each of them – given the cards are different, all suits are possible for each, so there are $4^2 = 16$ choices. Thus, overall,

$$|\mathcal{E}| = 16 \cdot 78 = 1248 .$$

Therefore,

$$\mathbb{P}[\mathcal{E}] = \frac{|\mathcal{E}|}{|\Omega|} = \frac{16 \cdot 78}{52 \cdot 51} = \frac{13 \cdot 3 \cdot 2^5}{13 \cdot 3 \cdot 2^2 \cdot 17} = \frac{8}{17} \approx 0.47 .$$

Task 8 – Robot Wears Socks

Suppose Joe is a k-legged robot, who wears a sock and a shoe on each leg. Suppose he puts on k socks and k shoes in some order, each equally likely. Each action is specified by saying whether he puts on a sock or a shoe, and saying which leg he puts it on. In how many ways can he put on his socks and shoes in a valid order? We say an ordering is valid if, for every leg, the sock gets put on before the shoe. Assume all socks are indistinguishable from each other, and all shoes are indistinguishable from each other.

First, note there are $2k$ objects — k shoes and k socks. Suppose we describe a sequence of actions, $Sock_1, Shoe_1, Sock_2, Shoe_2, \ldots, Sock_k, Shoe_k$.

This particular example means we first put a sock on leg 1, then a shoe on leg 1, then a sock on leg 2, etc. There are $(2k)!$ ways to order these actions. However, for each leg, there is only one valid ordering: the sock must come before the shoe. So we divide by 2^k and the total number of ways is $\frac{(2k)!}{2^k}$.

Alternatively, $\mathbb{P}(\text{valid ordering}) = \frac{|\text{valid orderings}|}{|\text{orderings}|}$, so $|\text{valid orderings}| = \mathbb{P}(\text{valid ordering}) * |\text{orderings}|$. We can compute $\mathbb{P}(\text{valid ordering}) = (1/2)^k$. Notice for any sequence of actions with each equally likely, the probability that the sock came before the shoe on a particular leg is $\frac{1}{2}$, so the probability this happened for each leg is $(1/2)^k$. Then $|\text{orderings}| = (2k)!$ because there are $2k$ actions that we can permute, all distinct. Multiplication gives the same answer as above.

Task 9 – Trick or Treat

Suppose on Halloween, someone is too lazy to keep answering the door, and leaves a jar of exactly N total candies. You count that there are exactly K of them which are kit kats (and the rest are not). The sign says to please take exactly n candies. Each item is equally likely to be drawn. Let X be the number of kit kats we draw (out of n). What is $\mathbb{P}(X = k)$, that is, the probability we draw exactly k kit kats?

$$\boxed{\mathbb{P}(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}}$$

We choose k out of the K kit kats, and $n - k$ out of the $N - K$ other candies. The denominator is the total number of ways to choose n candies out of N total.