CSE 312

Foundations of Computing II

Lecture 8: Linearity of Expectation

Last Class:

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Today:

- More Expectation Examples
- Linearity of Expectation
- Indicator Random Variables

Kandinsky

Review Random Variables

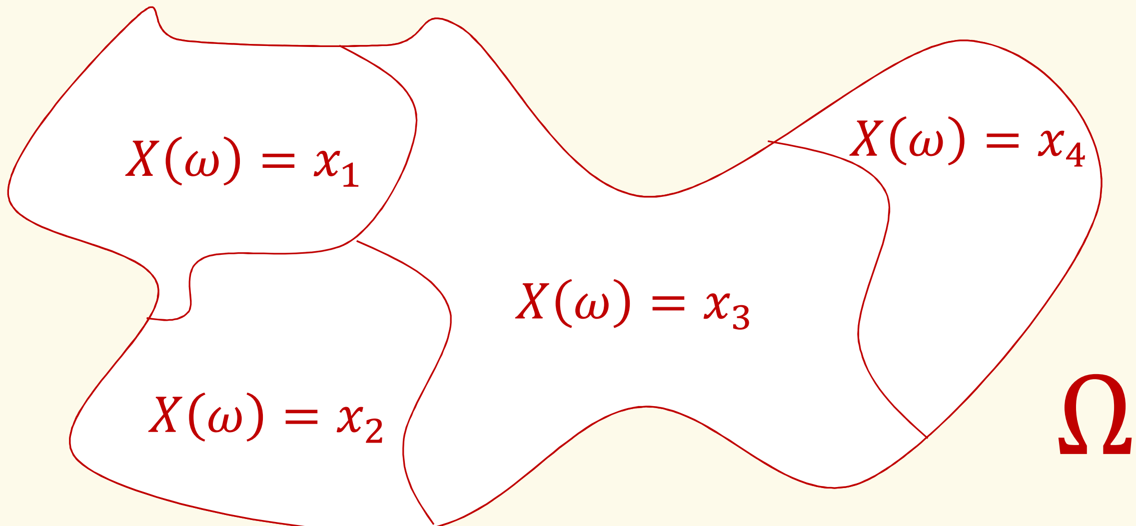
Definition. A **random variable (RV)** for a probability space (Ω, P) is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is its *range/support*: $X(\Omega)$ or Ω_X

$\{X = x_i\} = \{\omega \in \Omega \mid X(\omega) = x_i\}$

Random variables **partition** the sample space.

$\Sigma_{x \in X(\Omega)} P(X = x) = 1$



Review PMF and CDF

Definitions:

For a RV $X: \Omega \to \mathbb{R}$, the probability mass function (pmf) of X specifies, for any real number x, the probability that $X = x$

$$p_X(x) = P(X = x) = P(\{\omega \in \Omega \mid X(\omega) = x\})$$

$$\sum_{x \in \Omega_X} p_X(x) = 1$$

For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function (cdf) of X specifies, for any real number x, the probability that $X \leq x$

$$F_X(x) = P(X \leq x)$$

Review Expected Value of a Random Variable

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the **expectation** or **expected value** or **mean** of X is

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$$

or equivalently

$$\mathbb{E}[X] = \sum_{x \in X(\Omega)} x \cdot P(X = x) = \sum_{x \in \Omega_X} x \cdot p_X(x)$$

Intuition: "Weighted average" of the possible outcomes (weighted by probability)

Expectation

Example. Two fair coin flips

$\Omega = \{\mathrm{TT}, \mathrm{HT}, \mathrm{TH}, \mathrm{HH}\}$

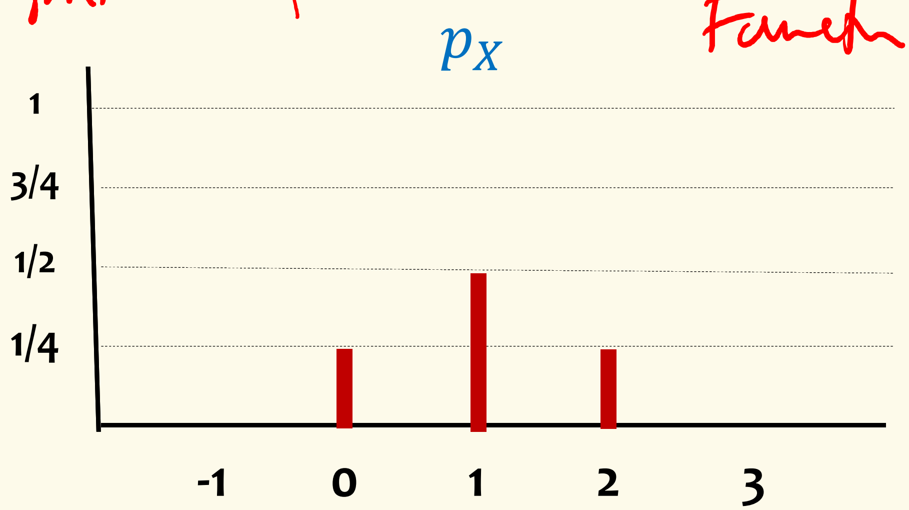
$X =$ number of heads

What is $\mathbb{E}[X]$?

pMF probability mass function

$0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

$x \cdot p_X(x)$



$$\mathbb{E}[X] = 0 \cdot p_X(0) + 1 \cdot p_X(1) + 2 \cdot p_X(2)$$

$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = \frac{1}{2} + \frac{1}{2} = 1$$

0

Another Interpretation

"If X is how much you win playing the game in one round. How much would you expect to win, on average, per game, when repeatedly playing?"

Answer: $\mathbb{E}[X]$

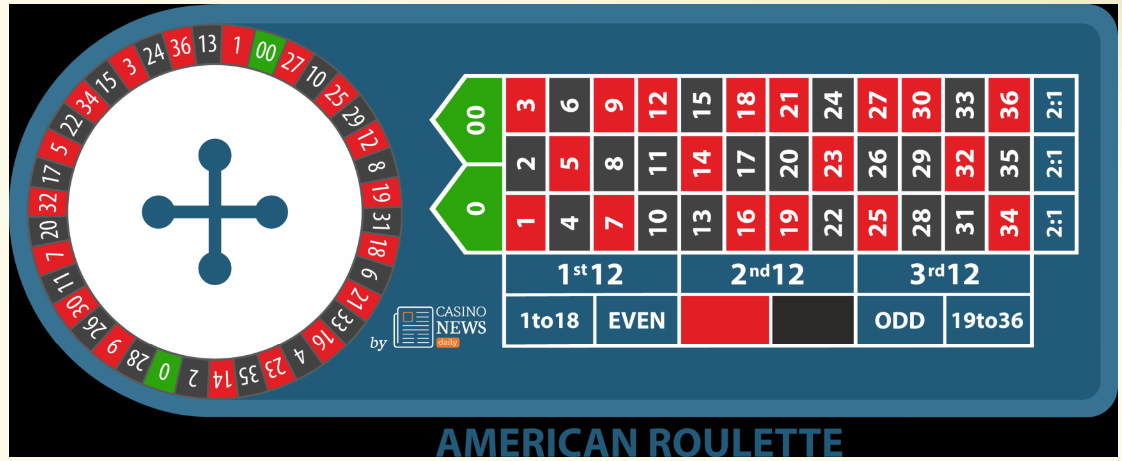
Roulette (USA)

Ω:

Numbers 1-36

- 18 Red
- 18 Black

Green 0 and 00



Note 0 and 00 are not EVEN

RVs for gains from some bets:

RV RED: If Red number turns up $+1$, if Black number, 0, or 00 turns up -1

$$\mathbb{E}[\text{RED}] = (+1) \cdot \frac{18}{38} + (-1) \cdot \frac{20}{38} = -\frac{2}{38} \approx -5.26\%$$

RV $1^{st}12$: If number 1-12 turns up $+2$, if number 13-36, 0, or 00 turns up -1

$$\mathbb{E}[1^{st}12] = (+2) \cdot \frac{12}{38} + (-1) \cdot \frac{26}{38} = -\frac{2}{38} \approx -5.26\%$$

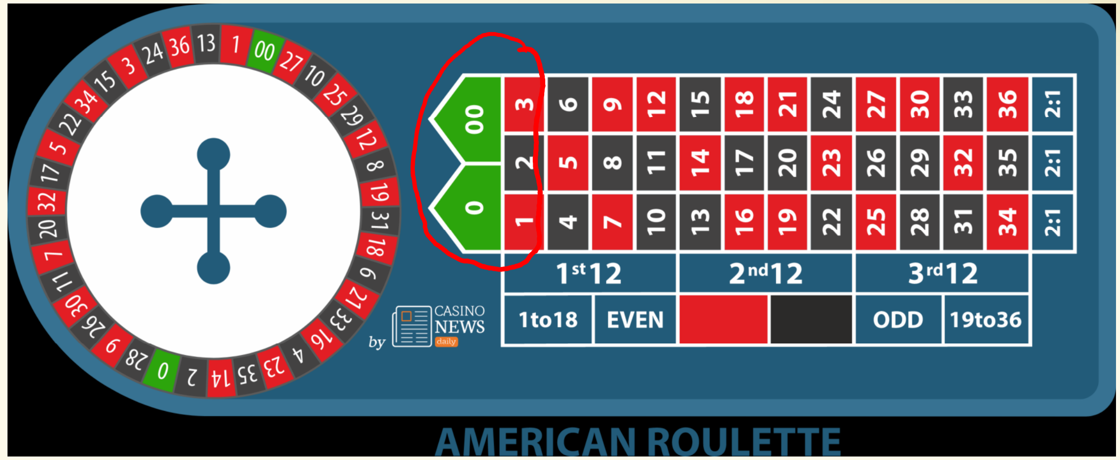
Roulette (USA)

Ω:

Numbers 1-36

- 18 Red
- 18 Black

Green 0 and 00



Note 0 and 00 are not EVEN

An even worse bet:

RV BASKET: If 0, 00, 1, 2, or 3 turns up +6 otherwise -1

$$\mathbb{E}[\text{BASKET}] = (+6)\cdot\frac{5}{38} + (-1)\cdot\frac{33}{38} = -\frac{3}{38} \approx -7.89\%$$

Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

$$\mathbb{E}[X] = 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6}$$

$$= 6 \cdot \frac{1}{6} = 1$$

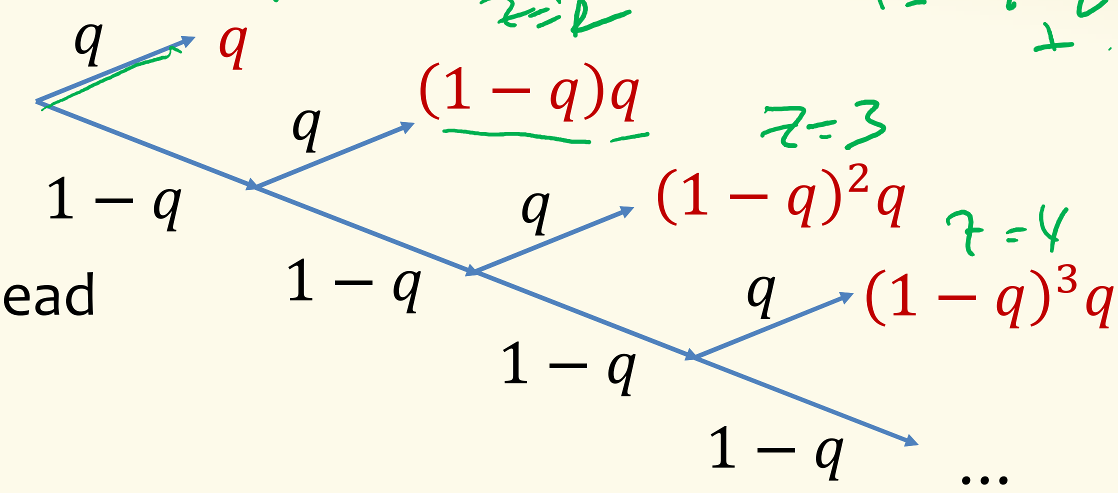
Example – Flipping a biased coin until you see heads

- Biased coin:

  $P(H) = q > 0$

  $P(T) = 1 - q$

- Z = # of coin flips until first head



$$P(Z = i) = q\,(1-q)^{i-1}$$

$$\mathbb{E}[Z] = \sum_{i=1}^{\infty} i \cdot P(Z = i) = \sum_{i=1}^{\infty} i \cdot q(1-q)^{i-1}$$

Converges, so $\mathbb{E}[Z]$ is finite

Can calculate this directly but…

Example – Flipping a biased coin until you see heads

- Biased coin:

  $P(H) = q > 0$

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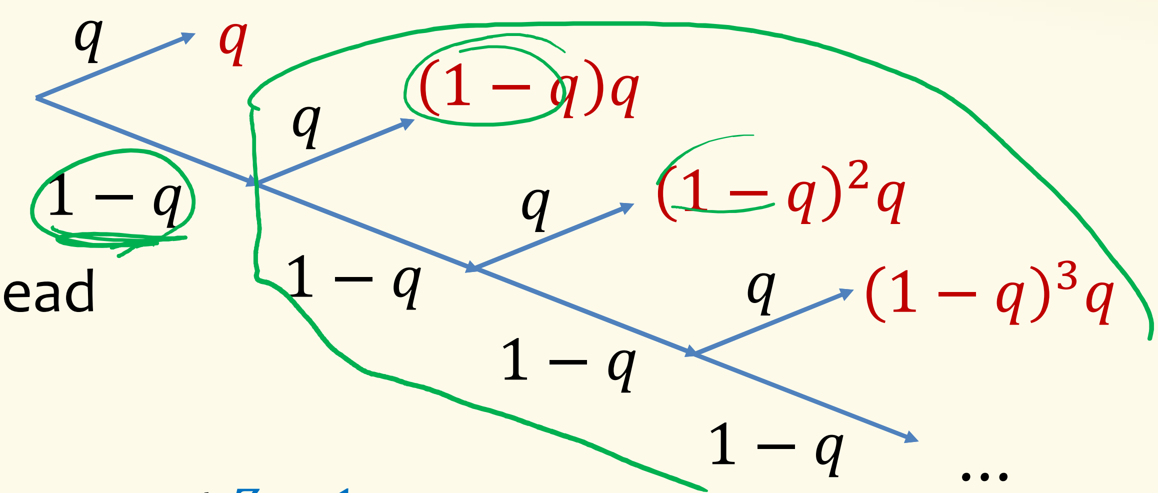


Another view: If you get heads first try you get $Z = 1$;
If you get tails you have used one try and have the same experiment left

E[# steps | success] = 1 — E[# steps | failed in step 1]

$$\mathbb{E}[Z] = q \cdot 1 + (1-q)(1 + \mathbb{E}(Z))$$

solve $q\,\mathbb{E}(Z) = q \cdot 1 + (1-q) \cdot 1 = 1$

Solving gives $q \cdot \mathbb{E}[Z] = q + (1-q) = 1$ Implies $\mathbb{E}[Z] = 1/q$

Expected Value of X = # of heads

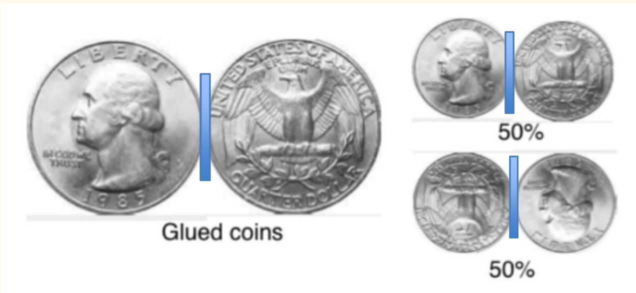
Each coin shows up heads half the time.

Two fair coins

$P(HT) = P(TH) = 0.25$
$P(HH) = P(TT) = 0.25$

$\mathbb{E}(X) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$

Glued coins



$P(HT) = P(TH) = 0.5$
$P(HH) = P(TT) = 0$

$\mathbb{E}(X) = 1 \cdot 1 = 1$

Attached coins

$P(HH) = P(TT) = 0.4$
$P(HT) = P(TH) = 0.1$

$\mathbb{E}(X) = 1 \cdot 0.2 + 2 \cdot 0.4 = 1$

Linearity of Expectation

Theorem. For any two random variables X and Y
(X, Y do not need to be independent)

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Or, more generally: For any random variables $X_1, \dots, X_n$,

$$\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

Because: $\mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[(X_1 + \cdots + X_{n-1}) + X_n]$

$$= \mathbb{E}[X_1 + \cdots + X_{n-1}] + \mathbb{E}[X_n] = \cdots$$

Linearity of Expectation – Proof

Theorem. For any two random variables X and Y

(X, Y do not need to be independent)

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

$f: \Omega \to \mathbb{R}$
$g: \Omega \to \mathbb{R}$
$(f+g): \Omega \to \mathbb{R}$
$h = f+g$

$$\mathbb{E}[X + Y] = \sum_\omega P(\omega)(X(\omega) + Y(\omega))$$

$$= \sum_\omega P(\omega)X(\omega) + \sum_\omega P(\omega)Y(\omega)$$

$$= \mathbb{E}[X] + \mathbb{E}[Y]$$

Example – Coin Tosses

We flip n coins, each one heads with probability p

Z is the number of heads, what is $\mathbb{E}(Z)$?

Example – Coin Tosses – The brute force method

We flip n coins, each one heads with probability p,

Z is the number of heads, what is $\mathbb{E}[Z]$?

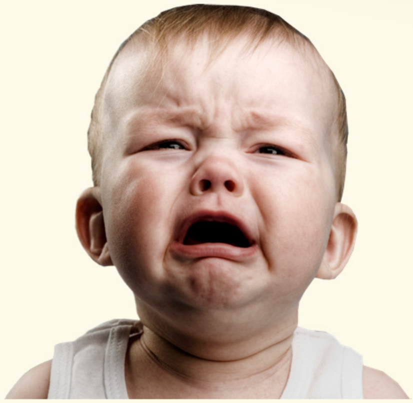
$$\mathbb{E}[Z] = \sum_{k=0}^{n} k \cdot P(Z = k) = \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} k \cdot \frac{n!}{k!\,(n-k)!} p^k (1-p)^{n-k} = \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!\,(n-k)!} p^{k-1} (1-p)^{n-k}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} p^k (1-p)^{(n-1)-k}$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{(n-1)-k} = np\big(p + (1-p)\big)^{n-1} = np \cdot 1 = np$$



Can we solve it more elegantly, please?

Computing complicated expectations

Often boils down to the following three steps:

- Decompose: Finding the right way to decompose the random variable into sum of simple random variables

$$X = X_1 + \cdots + X_n$$

- LOE: Apply linearity of expectation.

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n].$$

- Conquer: Compute the expectation of each X_i

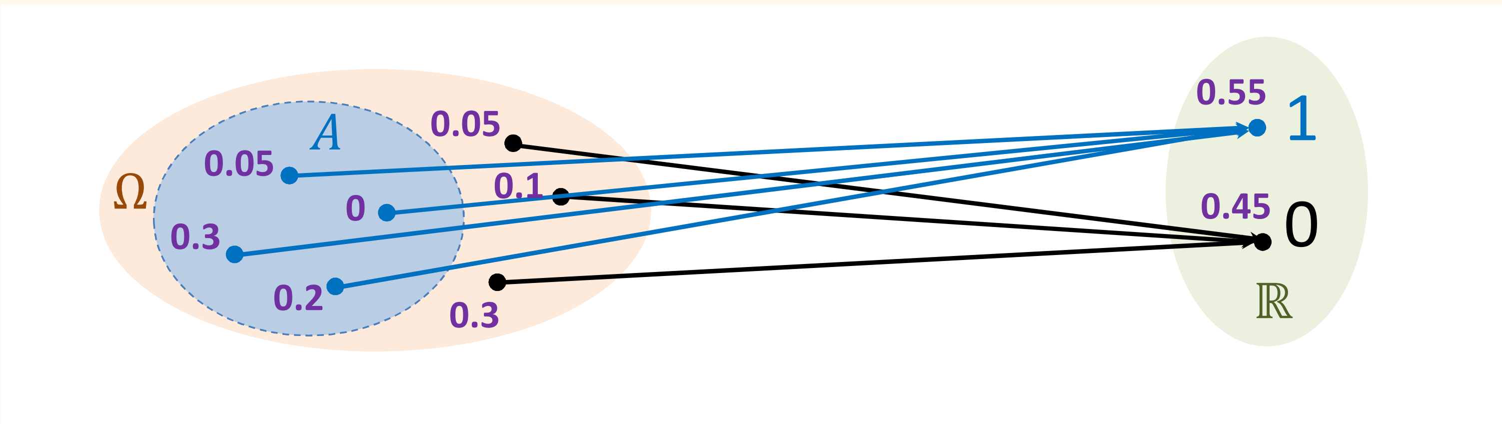
Often, X_i are indicator (0/1) random variables.

Indicator random variables

For any event A, can define the indicator random variable X_A for A

$$X_A = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{if event } A \text{ does not occur} \end{cases}$$

$$P(X_A = 1) = P(A)$$
$$P(X_A = 0) = 1 - P(A)$$



Example – Coin Tosses

We flip n coins, each one heads with probability p

Z is the number of heads, what is $\mathbb{E}[Z]$?

$\mathbb{E}[X_i] = p$

- $X_i = \begin{cases} 1, & i^{\text{th}} \text{ coin flip is heads} \\ 0, & i^{\text{th}} \text{ coin flip is tails.} \end{cases}$

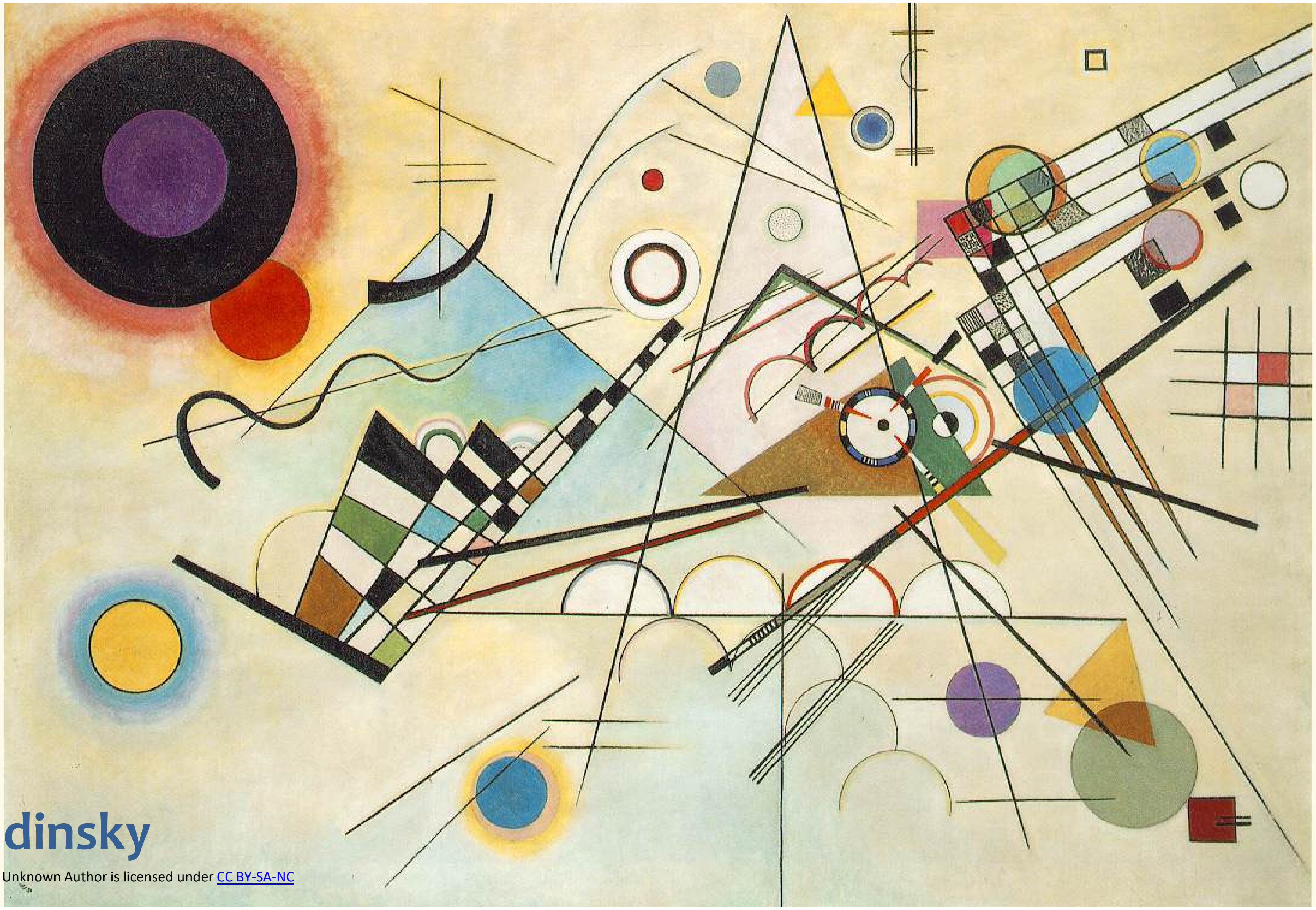
Fact. $Z = X_1 + \cdots + X_n$

Linearity of Expectation:

$$\mathbb{E}[Z] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] \quad = n \cdot p$$

$P(X_i = 1) = p$
$P(X_i = 0) = 1 - p$

$$\mathbb{E}[X_i] = p \cdot 1 + (1 - p) \cdot 0 = p$$



Kandinsky

Example: Returning Homeworks

- Class with n students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

What is $\mathbb{E}[X]$? Use linearity of expectation!

$\mathbf{Pr}(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

Decompose: What is X_i?

$X_i = 1$ iff i^{th} student gets own HW back

LOE: $\mathbb{E}[X] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n]$

Conquer: What is $\mathbb{E}[X_i]$? A. $\frac{1}{n}$ B. $\frac{1}{n-1}$ C. $\frac{1}{2}$

Poll: **pollev.com/paulbeame028**