

CSE 312

Foundations of Computing II

Lecture 4: Intro to Discrete Probability

Announcement

- PSet 1 due tonight
- PSet 2 posted this evening, due next Wednesday

Before probability a quick wrap-up for counting...

Final count: A less obvious consequence of the Binomial Theorem

$$0 = \sum_{k \in E} \binom{n}{k} - \sum_{k \in O} \binom{n}{k}$$

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

$= -1$ if k is odd
 $= +1$ if k is even

Corollary. For every n , if O and E are the sets of odd and even integers between 0 and n

odd size subset

$$\sum_{k \in O} \binom{n}{k} = \sum_{k \in E} \binom{n}{k}$$

even size subset

e.g., $n=4$:


$$\begin{matrix} 1 & 4 & 6 & 4 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{matrix}$$

Proof: Set $x = -1, y = 1$ in the binomial theorem

Summary of Counting: Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binary encoding/stars and bars
- Pigeonhole principle
- Combinatorial proofs
- Binomial Theorem

Agenda

- Events 
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- Another Example

Probability

- We want to model a process that is not deterministic.
 - i.e., outcome not determined a-priori
 - E.g. throwing dice, flipping a coin...
 - We want to numerically measure likelihood of outcomes = probability.
 - We want to make complex statements about these likelihoods.
- We will not argue why a certain physical process realizes the probabilistic model we study
 - Why is the outcome of the coin flip really “random”?
- First part of class: “Discrete” probability theory
 - Experiment with finite / discrete set of outcomes.
 - Will explore countably infinite and continuous outcomes later

Sample Space



Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Definition. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$
(i.e., E and F can't happen at same time)

Example:

- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$

Example: 4-sided Dice

Suppose I roll blue and red 4-sided dice. Let D_1 be the value of the blue die and D_2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$

B. $D_1 + D_2 = 6$

C. $D_1 = 2 * D_2$

		Die 2 (D_2)			
		1	2	3	4
Die 1 (D_1)	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice

Suppose I roll blue and red 4-sided dice. Let D_1 be the value of the blue die and D_2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$

$$A = \{(1,1), (1,2), (1,3), (1,4)\}$$

B. $D_1 + D_2 = 6$

$$B = \{(2,4), (3,3), (4,2)\}$$

C. $D_1 = 2 * D_2$

$$C = \{(2,1), (4,2)\}$$

Die 1 (D_1)

Die 2 (D_2)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice, Mutual Exclusivity

Are A and B mutually exclusive?

How about B and C ?

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	A & B	B & C
(a)	Yes	Yes
(b)	Yes	No
(c)	No	Yes
(d)	No	No

A. $D_1 = 1$

$$A = \{(1,1), (1,2), (1,3), (1,4)\}$$

B. $D_1 + D_2 = 6$

$$B = \{(2,4), (3,3), (4,2)\}$$

C. $D_1 = 2 * D_2$

$$C = \{(2,1), (4,2)\}$$

Die 1 (D_1)

Die 2 (D_2)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Agenda

- Events
- Probability ◀
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

$$\mathbb{P} : \Omega \rightarrow [0, 1]$$

that maps outcomes $\omega \in \Omega$ to probabilities $\mathbb{P}(\omega)$.

– Alternative notations: $\mathbb{P}(\omega) = P(\omega) = \text{Pr}(\omega)$

Most written formal CS, math, or stats uses \mathbb{P} or Pr but for slides we mostly use just P because it is easiest to read

Example – Coin Tossing

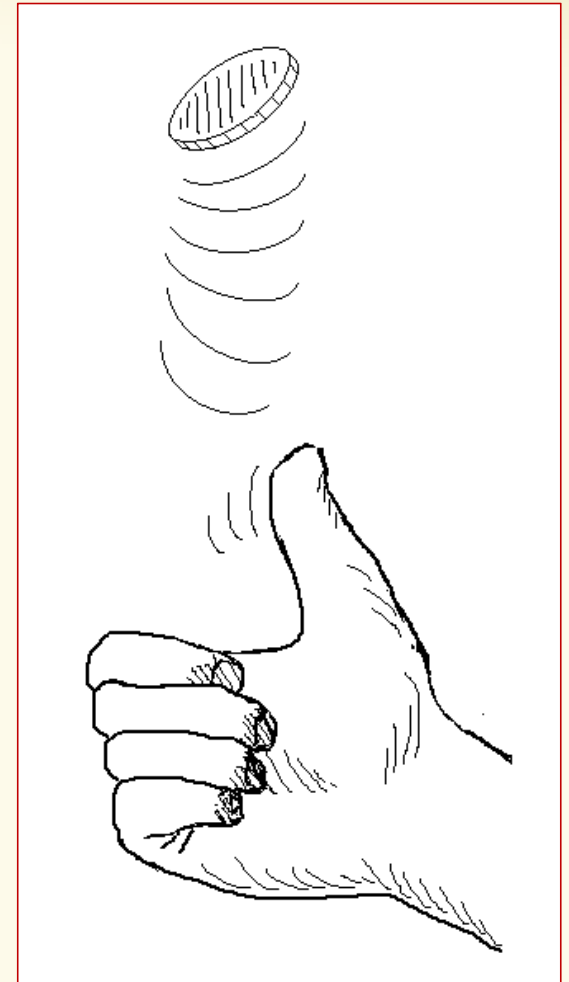
Imagine we toss one coin – outcome can be **heads** or **tails**.

$$\Omega = \{H, T\}$$

P ? Depends! What do we want to model?!

Fair coin toss

$$P(H) = P(T) = \frac{1}{2} = 0.5$$



Example – Coin Tossing

Imagine we toss one coin – outcome can be **heads** or **tails**.

$$\Omega = \{H, T\}$$

P ? Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$P(H) = 0.85, \quad P(T) = 0.15$$

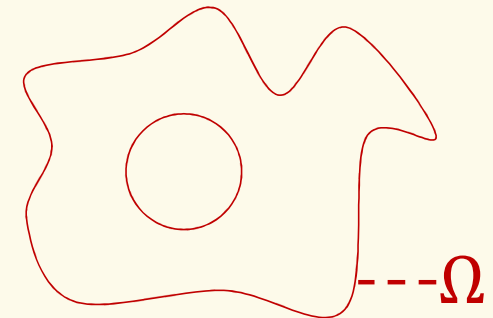
Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair (Ω, P) where:

- Ω is a set called the **sample space**.
- P is the **probability measure**, a function $P: \Omega \rightarrow [0,1]$ such that:
 - $P(x) \geq 0$ for all $x \in \Omega$
 - $\sum_{x \in \Omega} P(x) = 1$

Set of possible elementary outcomes



Specify Likelihood (or probability) of each elementary outcome

sample space

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Uniform Probability Space

Definition. A uniform probability space is a pair (Ω, P) such that

$$P(x) = \frac{1}{|\Omega|}$$

for all $x \in \Omega$.

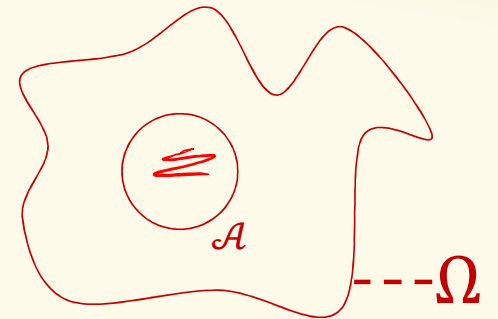
Examples:

- Fair coin $P(x) = \frac{1}{2}$
- Fair 6-sided die $P(x) = \frac{1}{6}$

Events

Definition. An **event** in a probability space (Ω, P) is a subset $\mathcal{A} \subseteq \Omega$. Its probability is

$$P(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} P(\omega)$$



Abuse of notation: When the event \mathcal{A} is a set $\{\omega\}$ with just one outcome ω we write

$$P(\omega) \text{ instead of } P(\{\omega\})$$

But that is OK, because they are equal by definition.

Don't care if the argument is an event or outcome!

Agenda

- Events
- Probability
- **Equally Likely Outcomes** ◀
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Example: 4-sided Dice, Event Probability

Think back to 4-sided die. Suppose each die is fair.
What is the probability of event B ? $P(B) = ???$

$$B. D1 + D2 = 6$$

$$B = \{(2,4), (3,3), (4,2)\}$$

$$P(B) = 3/16$$

$$\frac{3}{16}$$

Die 1 (D1)

Die 2 (D2)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Equally Likely Outcomes

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$,

$$P(E) = \frac{|E|}{|\Omega|}$$

$$\begin{aligned} & \sum_{\omega \in E} P(\omega) \\ &= \sum_{\omega \in E} \frac{1}{|\Omega|} \\ &= \frac{|E|}{|\Omega|} \end{aligned}$$

This follows from the definitions of the probability of an event and uniform probability spaces.

Example – Coin Tossing

Toss a coin **100** times. Each outcome is **equally likely** (and assume the outcome of one toss does not impact another).

What is the probability of seeing **50** heads?

(a) $\frac{1}{2}$

(b) $\frac{1}{2^{50}}$

(c) $\frac{\binom{100}{50}}{2^{100}}$

(d) Not sure



100 coins
 2^{100} possible: ~~HT HT HT~~ HT
| | | |
Some 50 are heads

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Brain Break



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- Events
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- Equally Likely Outcomes
- **Probability Axioms and Beyond Equally Likely Outcomes** ◀
- More Examples

Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this applies to *any* probability space (not just uniform)

Axiom 1 (Non-negativity): $P(E) \geq 0$.

Axiom 2 (Normalization): $P(\Omega) = 1$.

Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.

Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$.

Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

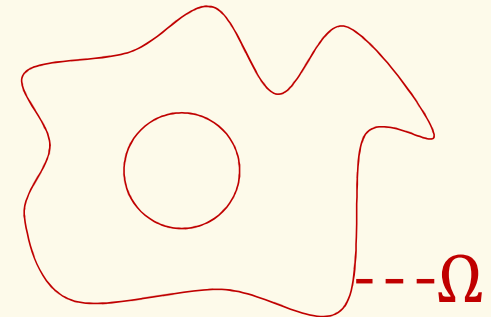
Review Probability space

Either finite or infinite countable (e.g., integers)

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 - $\sum_{x \in \Omega} P(x) = 1$

Set of possible elementary outcomes



Specify Likelihood (or probability) of each elementary outcome

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Axioms of Probability

Let (Ω, P) be a probability space. Then, the following properties hold for any events $E, F \subseteq \Omega$.

Axiom 1 (Non-negativity): $P(E) \geq 0$. ←

Axiom 2 (Normalization): $P(\Omega) = 1$. ←

Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Called “axioms” because all properties of P follow from them!

complement of event E E^c

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.

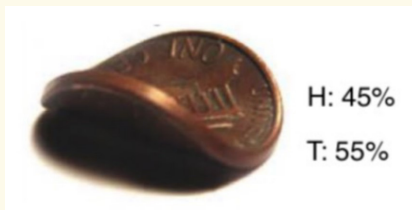
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$.

Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

Non-equally Likely Outcomes

Many probability spaces can have **non-equally likely outcomes**.

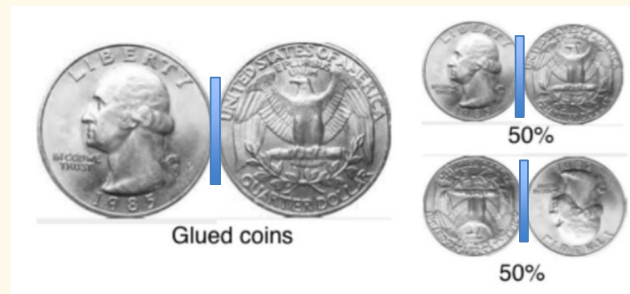
Biased coin



$$P(H) = p$$

$$P(T) = 1 - p$$

Glued coins



$$P(\underline{HT}) = P(\underline{TH}) = 0.5$$

$$P(\underline{HH}) = P(\underline{TT}) = 0$$

Attached coins



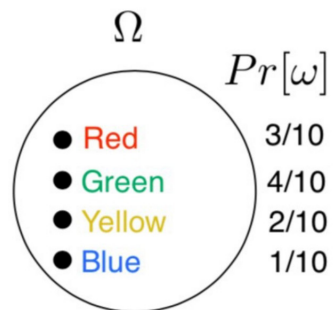
$$P(\underline{HH}) = P(\underline{TT}) = 0.4$$

$$P(\underline{HT}) = P(\underline{TH}) = 0.1$$

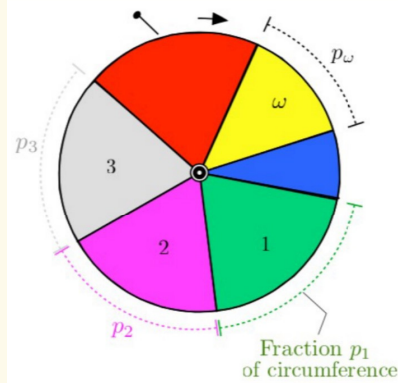
More Examples of Non-equally Likely Outcomes



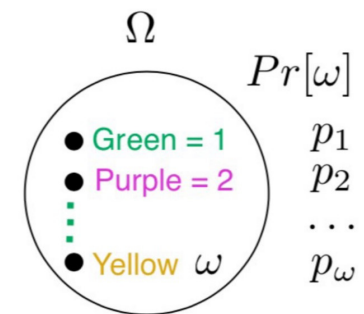
Physical experiment



Probability model



Physical experiment



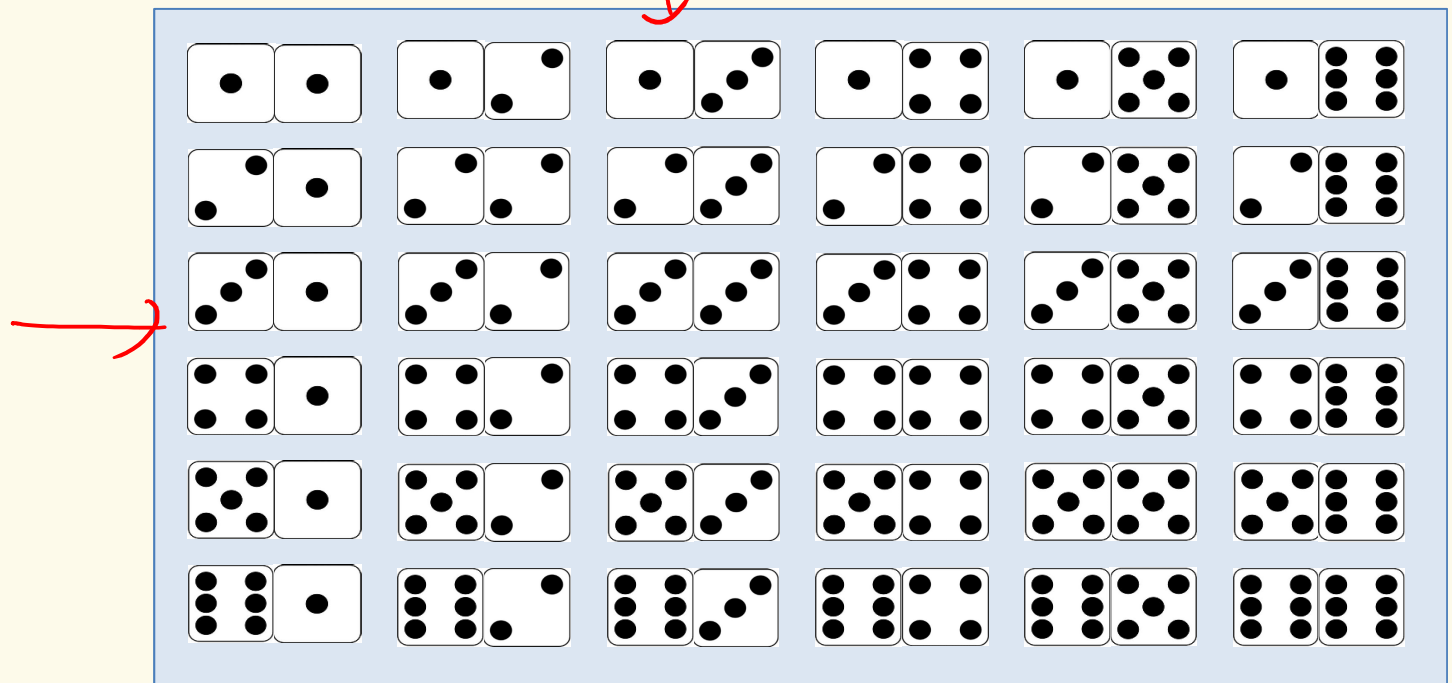
Probability model

Agenda

- Events
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- **Another Example (Equally Likely)** ◀

Example: Dice Rolls

Suppose I had two, fair, 6-sided dice that we roll once each.
What is the probability that we see *at least one 3 in the two rolls?*



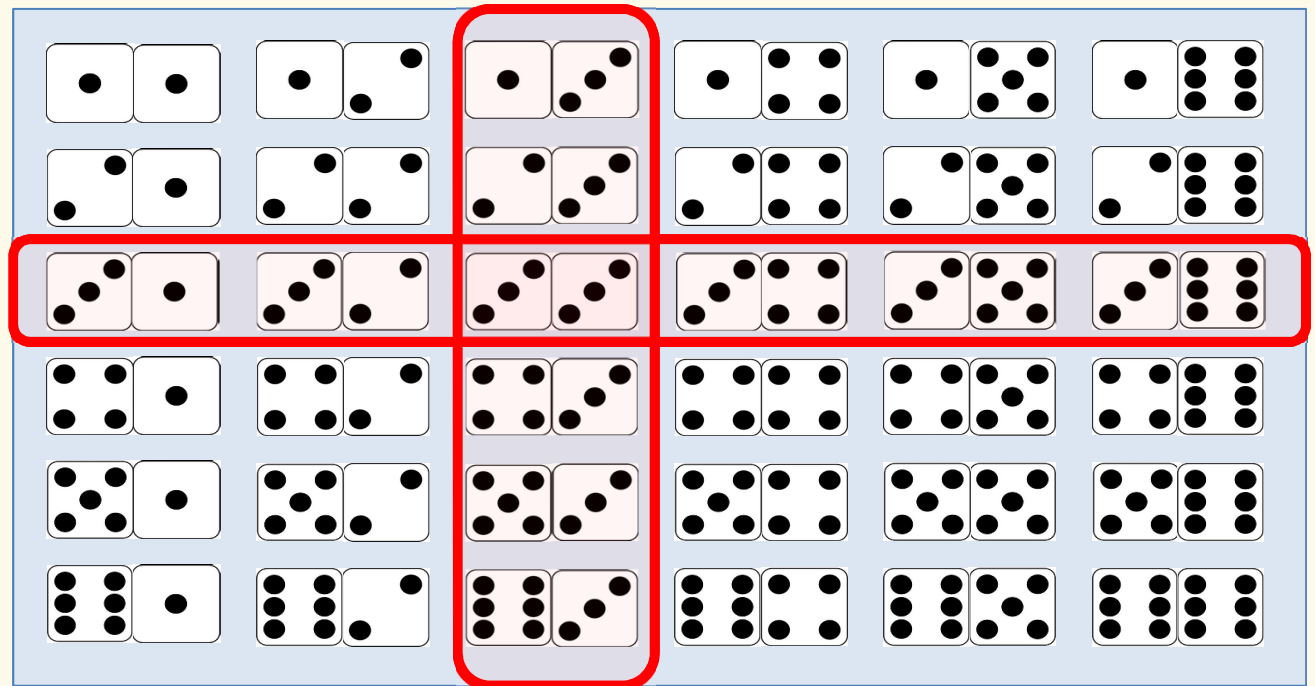
Example: Dice Rolls

Suppose I had two, fair, 6-sided dice that we roll once each.
What is the probability that we see *at least one 3 in the two rolls*?

Event has
 $6 + 6 - 1 = 11$
outcomes

$$|\Omega| = 36$$

$$P(\geq \text{one } 3) = \frac{11}{36}$$



Fun: Von Neumann's Trick with a biased coin

- How to use a biased coin to get a fair coin flip:
 - Suppose that you have a biased coin:
 - $P(H) = p$ $P(T) = 1 - p$

1. Flip coin twice: If you get HH or TT go to step 1
2. If you got HT output H; if you got TH output T.

Why is it fair? $P(HT) = p(1 - p) = (1 - p)p = P(TH)$

Drawback: You may never get to step 2.