Quiz Section 8 – Solutions

Review

1) Markov's Inequality: Let X be a non-negative random variable, and $\alpha > 0$. Then, $\mathbb{P}(X \geqslant \alpha) \leqslant \frac{\mathbb{E}[X]}{\alpha}$.

2) Chebyshev's Inequality: Suppose Y is a random variable with $\mathbb{E}[Y] = \mu$ and $\text{Var}(Y) = \sigma^2$. Then, for any $\alpha > 0$, $\mathbb{P}(|Y - \mu| \geqslant \alpha) \leqslant \frac{\sigma^2}{\alpha^2}$.

3) Chernoff Bound: Suppose $X = X_1 + \cdots + X_n$ where the X_i are independent and in $[0, 1]$. Let $\mu = \mathbb{E}[X]$. Then, for any $0 < \delta \leqslant 1$, $\mathbb{P}(|X - \mu| \geqslant \delta\mu) \leqslant e^{-\frac{\delta^2\mu}{4}}$ and for any $\delta > 0$, $\mathbb{P}(X - \mu \geqslant \delta\mu) \leqslant e^{-\delta^2\mu/4}$.

4) Multivariate: Discrete to Continuous:

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$	$f_{X,Y}(x,y) \neq \mathbb{P}(X = x, Y = y)$
Joint range/support $\Omega_{X,Y}$	$\{(x,y) \in \Omega_X \times \Omega_Y : p_{X,Y}(x,y) > 0\}$	$\{(x,y) \in \Omega_X \times \Omega_Y : f_{X,Y}(x,y) > 0\}$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \leqslant x, s \leqslant y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s)\, ds dt$
Normalization	$\sum_{x,y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_y p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y) p_{X,Y}(x,y)$	$\mathbb{E}[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Independence must have	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\Omega_{X,Y} = \Omega_X \times \Omega_Y$
Conditional PMF/PDF	$p_{X\|Y}(x\|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\|Y}(x\|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$\mathbb{E}[X\|Y = y] = \sum_x x \cdot p_{X\|Y}(x\|y)$	$\mathbb{E}[X\|Y = y] = \int_{-\infty}^{\infty} x f_{X\|Y}(x\|y) dx$

Task 1 – A Dysfunctional Family

Rick and his grandson Morty are set to meet at a certain time. Since their relationship is a little strained, neither of them wants to be there on time. Let $X \sim Unif(0, 10)$ be the amount of minutes Morty is going to be late. Rick has cameras around the meeting spot and will observe Morty's arrival time $X = x$. Then, he will arrive at the meeting spot $Unif(x, 5x)$ minutes late. Let Y be the random variable indicating how late Rick will be.

a) Using the above definitions determine f_X, $f_{Y|X}$, and f_{XY}. (You will want to determine f_{YX} and use it to determine f_{XY}.).

Since X is a uniform RV on $(0, 10)$, we have

$$f_X(x) = \begin{cases} \frac{1}{10} & x \in (0, 10) \\ 0 & \text{otherwise} \end{cases}$$

Also given that $X = x$, Y is also uniform on $(x, 5x)$ so

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{4x} & y \in (x, 5x) \\ 0 & \text{otherwise} \end{cases}$$

Since $f_{Y|X}(y|x) = \frac{f_{YX}(y,x)}{f_X(x)}$, we have

$$f_{YX}(y,x) = f_{Y|X}(y|x) \cdot f_X(x) = \begin{cases} \frac{1}{40x} & x \in (0, 10) \text{ and } y \in (x, 5x) \\ 0 & \text{otherwise} \end{cases}$$

Then

$$f_{XY}(x,y) = f_{YX}(y,x) = \begin{cases} \frac{1}{40x} & x \in (0,10) \text{ and } y \in (x,5x) \\ 0 & \text{otherwise} \end{cases}$$

b) Compute $\mathbb{E}[Y]$.

By definition since Y conditioned on $X = x$ is a uniform RV on $(x, 5x)$, we have

$$\mathbb{E}[Y \mid X = x] = \frac{x + 5x}{2} = 3x.$$

(Alternatively, we could compute it from first principles using $f_{Y|X}$ and the definition

$$\mathbb{E}[Y|X = x] = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|x)dy = \int_{x}^{5x} \frac{y}{4x}dy = \frac{y^2}{8x}\Big|_{y=x}^{y=5x} = \frac{25x^2}{8x} - \frac{x^2}{8x} = \frac{24x}{8} = 3x.)$$

Then, using the Law of Total Expectation we get:

$$\mathbb{E}[Y] = \int_{-\infty}^{\infty} \mathbb{E}[Y|X = x] \cdot f_X(x)dx = \int_0^{10} \frac{3x}{10}dx = \frac{3}{10} \cdot \frac{x^2}{2}\Big|_0^{10} = \frac{300}{20} - 0 = 15$$

Note: This is a place where the Law of Total Expectation makes things **much** easier than figuring out the PDF of Y and doing direct calculation of the expectation:
Just to see how bad it would get... by definition we have:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_0^{10} f_{XY}(x,y)dx$$

since we know that $f_{XY}(x,y)$ is 0 when $x \leqslant 0$ or $x \geqslant 10$. However, we can't just plug in $\frac{1}{40x}$ for $f_{XY}(x,y)$ because we also need to satisfy that $x \leqslant y \leqslant 5x$ for that value to be correct. For a fixed value of y, which values of x could work? We need to have $x \leqslant y$, but we also need to have $y \leqslant 5x$ or in other worse $x \geqslant y/5$. In particular, this is equivalent to $y/5 \leqslant x \leqslant y$. Therefore we need $\max(0, y/5) \leqslant x \leqslant \min(10, y)$. Therefore

$$f_Y(y) = \int_0^{10} f_{XY}(x,y)dx = \int_{\max(0,y/5)}^{\min(10,y)} \frac{1}{40x}dx.$$

This would have non-zero contributions for all y with $0 \leqslant y \leqslant 50$ and would be a big mess to calculate since the integral involves $1/x$ which would have logarithms in it...

Task 2 – Tail bounds

Suppose $X \sim$ Binomial$(6, 0.4)$. We will bound $\mathbb{P}(X \geqslant 4)$ using the tail bounds we've learned, and compare this to the true result.

a) Give an upper bound for this probability using Markov's inequality. Why can we use Markov's inequality?

> We know that the expected value of a binomial distribution is np, so: $\mathbb{P}(X \geqslant 4) \leqslant \frac{\mathbb{E}[X]}{4} = \frac{2.4}{4} = 0.6$. We can use it since X is nonnegative.

b) Give an upper bound for this probability using Chebyshev's inequality. You may have to rearrange algebraically and it may result in a weaker bound.

> $\mathbb{P}(X \geqslant 4) = \mathbb{P}(X - 2.4 \geqslant 1.6) \leqslant \mathbb{P}(|X - 2.4| \geqslant 1.6)$ we can add those absolute value signs because that only adds more possible values, so it is an upper bound on the probability of $X - 2.4 \geqslant 1.6$. Then, using Chebyshev's inequality we get:
> $\mathbb{P}(|X - 2.4| \geqslant 1.6) \leqslant \frac{Var(X)}{1.6^2} = \frac{1.44}{1.6^2} = 0.5625$

c) Give an upper bound for this probability using the Chernoff bound.

> $\mathbb{P}(X \geqslant 4) = \mathbb{P}(X \geqslant (1 + \frac{2}{3})2.4) \leqslant e^{-(\frac{2}{3})^2 \mathbb{E}[X]/4} = e^{-4 \times 2.4/36} \approx 0.77$

d) Give the exact probability.

> Since X is a binomial, we know it has a range from 0 to n (or in this case 0 to 6). Thus, the possible values to satisfy $X \geqslant 4$ are 4, 5, or 6. We plug in the PMF for each to get: $\mathbb{P}(X \geqslant 4) = \mathbb{P}(X = 4) + \mathbb{P}(X = 5) + \mathbb{P}(X = 6) = \binom{6}{4}(0.4)^4(0.6)^2 + \binom{6}{5}(0.4)^5(0.6) + \binom{6}{6}0.4^6 \approx 0.1792$

Task 3 – Exponential Tail Bounds

Let $X \sim$ Exp(λ) and $k > 1/\lambda$. Recall that $\mathbb{E}[X] = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$.

a) Use Markov's inequality to bound P$(X \geqslant k)$.

$$\mathbb{P}(X \geqslant k) \leqslant \frac{1}{\lambda k}$$

b) Use Chebyshev's inequality to bound P$(X \geqslant k)$.

$$\mathbb{P}(X \geqslant k) = \mathbb{P}\left(X - \frac{1}{\lambda} \geqslant k - \frac{1}{\lambda}\right) \leqslant \mathbb{P}\left(\left|X - \frac{1}{\lambda}\right| \geqslant k - \frac{1}{\lambda}\right) \leqslant \frac{1}{\lambda^2(k - 1/\lambda)^2} = \frac{1}{(\lambda k - 1)^2}$$

c) What is the exact formula for P$(X \geqslant k)$?

$$\mathbb{P}(X \geqslant k) = e^{-\lambda k}$$

d) For $\lambda k \geqslant 3$, how do the bounds given in parts (a), (b), and (c) compare?

$$e^{-\lambda k} < \frac{1}{(\lambda k - 1)^2} < \frac{1}{\lambda k}$$

> so Markov's inequality gives the worst bound.

Task 4 – How many samples?

Let $X = X_1 + \dots X_n$ be the sum of n independent $Poisson(\lambda)$ random variables. Recall that the Poisson distribution has expectation and variance both equal to λ and has the summation property that X is a $Poisson(n\lambda)$ random variable.

a) How large a value of n would Chebyshev's inequality need to guarantee that $\mathbb{P}(X \leqslant \mathbb{E}[X]/2) \leqslant 0.01$?

We have

$$\mathbb{P}(X \leqslant \mathbb{E}[X]/2) = \mathbb{P}(X - \mathbb{E}[X] \leqslant -\mathbb{E}[X]/2) \leqslant \mathbb{P}(|X - \mathbb{E}[X]| \geqslant \mathbb{E}[X]/2).$$

Applying Chebyshev's inequality we have

$$\mathbb{P}(X \leqslant \mathbb{E}[X]/2) \leqslant \mathbb{P}(|X - \mathbb{E}[X]| \geqslant \mathbb{E}[X]/2) \leqslant \frac{4\mathrm{Var}\,(X)}{\mathbb{E}[X]^2} = \frac{4n\lambda}{n^2\lambda^2} = \frac{4}{n\lambda}.$$

In order for this to be at most 0.01, we require $\boxed{n \geqslant 400/\lambda}$.

b) How large a value of n would Markov's inequality need to guarantee that $\mathbb{P}(X \leqslant \mathbb{E}[X]/2) \leqslant 0.01$?

X is non-negative so Markov's inequality applies to X, but no value of n will guarantee any probability less than 1.