Quiz Section 4

Review

1) **Probability Mass.** For every random variable $X$, we have $\sum_x P(X = x) = \underline{1}.$

2) **Expectation.** $E[X] = \underline{1}.$

3) **Linearity of expectation.** For any random variables $X_1, \ldots, X_n$, and real numbers $a_1, \ldots, a_n,$

$$E[a_1 X_1 + \cdots + a_n X_n] = \underline{1}.$$  

4) **Variance.** $\text{Var}(X) = E[(X - E[X])^2] = E[X^2] - E[X]^2.$ $\text{Var}(aX + b) = \underline{1} \text{Var}(X).$

5) **Independence.** Two random variables $X$ and $Y$ are independent if ________________.

6) **Variance and Independence.** For any two independent random variables $X$ and $Y$, $\text{Var}(X + Y) = \underline{1}.$

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**Task 1 – Identify that range!**

Identify the support/range $\Omega_X$ of the random variable $X$, if $X$ is...

- a) The sum of two rolls of a six-sided die.
- b) The number of lottery tickets I buy until I win it.
- c) The number of heads in $n$ flips of a coin with $0 < P(\text{head}) < 1$.
- d) The number of heads in $n$ flips of a coin with $P(\text{head}) = 1$.

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**Task 2 – Symmetric Difference**

Suppose $A$ and $B$ are random, independent (possibly empty) subsets of $\{1, 2, \ldots, n\}$, where each subset is equally likely to be chosen as $A$ or $B$. Consider $A \Delta B = (A \cap B^C) \cup (B \cap A^C) = (A \cup B) \cap (A^C \cup B^C)$, i.e., the set containing elements that are in exactly one of $A$ and $B$. Let $X$ be the random variable that is the size of $A \Delta B$. What is $E[X]$?

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**Task 3 – Hungry Washing Machine**

You have 10 pairs of socks (so 20 socks in total), with each pair being a different color. You put them in the washing machine, but the washing machine eats 4 of the socks chosen at random. Every subset of 4 socks is equally probable to be the subset that gets eaten. Let $X$ be the number of complete pairs of socks that you have left.

- a) What is the range of $X$, $\Omega_X$ (the set of possible values it can take on)? What is the probability mass function of $X$?
- b) Find $E[X]$ from the definition of expectation.
- c) Find $E[X]$ using linearity of expectation.
- d) Which way was easier? Doing both (a) and (b), or just (c)?
Task 4 – Balls in Bins

Let $X$ be the number of bins that remain empty when $m$ balls are distributed into $n$ bins randomly and independently. For each ball, each bin has an equal probability of being chosen. (Notice that two bins being empty are not independent events: if one bin is empty, that decreases the probability that the second bin will also be empty. This is particularly obvious when $n = 2$ and $m > 0$.) Find $E[X]$.

Task 5 – Frogger

A frog starts on a 1-dimensional number line at 0. At each second, independently, the frog takes a unit step right with probability $p_1$, to the left with probability $p_2$, and doesn’t move with probability $p_3$, where $p_1 + p_2 + p_3 = 1$. After 2 seconds, let $X$ be the location of the frog.

a) Find $p_X(k)$, the probability mass function for $X$.

b) Compute $E[X]$ from the definition.

c) Compute $E[X]$ again, but using linearity of expectation.

Task 6 – 3-sided Die

Let the random variable $X$ be the sum of two independent rolls of a fair 3-sided die. (If you are having trouble imagining what that looks like, you can use a 6-sided die and change the numbers on 3 of its faces.)

a) What is the probability mass function of $X$?

b) Find $E[X]$ directly from the definition of expectation.

c) Find $E[X]$ again, but this time using linearity of expectation.

d) What is $Var(X)$?

Task 7 – Practice

a) Let $X$ be a random variable with $p_X(k) = ck$ for $k \in \{1, \ldots, 5\} = \Omega_X$, and 0 otherwise. Find the value of $c$ that makes $X$ follow a valid probability distribution and compute its mean and variance ($E[X]$ and $Var(X)$).

b) Let $X$ be any random variable with mean $E[X] = \mu$ and variance $Var(X) = \sigma^2$. Find the mean and variance of $Z = \frac{X - \mu}{\sigma}$. (When you’re done, you’ll see why we call this a “standardized” version of $X$!)

c) Let $X, Y$ be independent random variables. Find the mean and variance of $X - 3Y - 5$ in terms of $E[X], E[Y], Var(X)$, and $Var(Y)$.

d) Let $X_1, \ldots, X_n$ be independent and identically distributed (iid) random variables each with mean $\mu$ and variance $\sigma^2$. The sample mean is $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$. Find the mean and variance of $\bar{X}$. If you use the independence assumption anywhere, explicitly label at which step(s) it is necessary for your equalities to be true.

Task 8 – Expectations, Independence, and Variance

a) Let $U$ be a random variable which is uniform over the set $[n] = \{1, 2, \ldots, n\}$, i.e., $P(U = i) = \frac{1}{n}$ for all $i \in [n]$. Compute $E[U^2]$ and $Var(U)$.

Hint: $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ and $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

b) Let $Y_1$ and $Y_2$ be the independent outcomes of two dice rolls, and let $Z = Y_1 + Y_2$. Then, compute $E[Z^2]$ and $Var(Z)$.

Hint: Try to use an indirect solution using linearity and independence, without the need of explicitly giving the distribution of $Z^2$. 