Quiz Section 3

Review

1) Conditional Probability. $\mathbb{P}(\mathcal{B} \mid \mathcal{A}) =$ ____________

2) Independent Events. Two events $\mathcal{A}, \mathcal{B}$ are **independent** if $\mathbb{P}(\mathcal{A} \cap \mathcal{B}) =$ ____________.
If $\mathbb{P}(\mathcal{A}) \neq 0$, this is equivalent to $\mathbb{P}(\mathcal{B} \mid \mathcal{A}) =$ ____________.
If $\mathbb{P}(\mathcal{B}) \neq 0$, this is equivalent to $\mathbb{P}(\mathcal{A} \mid \mathcal{B}) =$ ____________.

3) Partition. Nonempty events $\mathcal{E}_1, \ldots, \mathcal{E}_n$ partition the sample space Ω iff

(1) ________________________, (2) ________________________

4) Bayes Rule. For any events $\mathcal{A}$ and $\mathcal{B}$, $\mathbb{P}(\mathcal{A}|\mathcal{B}) =$ _______________ .

5) Chain Rule: Suppose $\mathcal{A}_1, ..., \mathcal{A}_n$ are events. Then,

$$\mathbb{P}(\mathcal{A}_1 \cap ... \cap \mathcal{A}_n) = \underline{\hspace{3cm}}$$

6) Law of Total Probability (LTP): Suppose $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is a partition of Ω and let $\mathcal{B}$ be any event. Then $\mathbb{P}(\mathcal{B}) = \sum_{i=1}^{n} \mathbb{P}(\mathcal{B} \cap \mathcal{E}_i) =$ ____________

7) Bayes Theorem with LTP: $\mathcal{E}_1, \ldots, \mathcal{E}_n$ is a partition of Ω and let $\mathcal{B}$ be any event. Then

$$\mathbb{P}(\mathcal{E}_1 \mid \mathcal{B}) = \frac{\underline{\hspace{3cm}}}{\sum_{i=1}^{n} \underline{\hspace{3cm}}}.$$

8) Probability Mass. For every random variable X, we have $\sum_{x \in \Omega(X)} \mathbb{P}(X = x) =$ _______.

9) Expectation. $\mathbb{E}[X] =$ _______.

Task 1 – Naive Bayes

Most of Section 3 will be an introduction to an application of Bayes' Theorem called the Naive Bayes Classifier.

Task 2 – Flipping Coins

We consider two independent tosses of the same coin. The coin is "heads" one quarter of the time.

a) What is the probability that the second toss is "heads" given that the first toss is "tails"?

b) What is the probability that the second toss is "heads" given that at least one of the tosses is "tails"?

c) In the probability space of this task, give an example of two events that are disjoint but not independent.

d) In the probability space of this task, give an example of two events that are independent but not disjoint.

Task 3 – Balls from an Urn – Take 2

Say an urn contains three red balls and four blue balls. Imagine we draw three balls without replacement. (You can assume every ball is uniformly selected among those remaining in the urn.)

a) What is the probability that all three balls are all of the same color?

b) What is the probability that we get more than one red ball given the first ball is red?

Task 4 – Game Show

Corrupted by their power, the judges running the popular game show America's Next Top Mathematician have been taking bribes from many of the contestants. During each of two episodes, a given contestant is either allowed to stay on the show or is kicked off. If the contestant has been bribing the judges, they will be allowed to stay with probability 1. If the contestant has not been bribing the judges, they will be allowed to stay with probability 1/3, independent of what happens in earlier episodes. Suppose that 1/4 of the contestants have been bribing the judges. The same contestants bribe the judges in both rounds.

a) If you pick a random contestant, what is the probability that they are allowed to stay during the first episode?

b) If you pick a random contestant, what is the probability they are allowed to stay during both episodes?

c) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that they get kicked off during the second episode?

d) If you pick a random contestant who was allowed to stay during the first episode, what is the probability that they were bribing the judges?

Task 5 – Coins

There are three coins, C_1, C_2, and C_3. The probability of "heads" is 1 for C_1, 0 for C_2, and p for C_3. A coin is picked among these three uniformly at random, and then flipped a certain number of times.

a) What is the probability that the first n flips are tails?

b) Given that the first n flips were tails, what is the probability that C_1 was flipped / C_2 was flipped / C_3 was flipped?

Task 6 – Parallel Systems

A parallel system functions whenever at least one of its components works. Consider a parallel system of n components and suppose that each component works with probability p independently.

a) What is the probability the system is functioning?

b) If the system is functioning, what is the probability that component 1 is working?

c) If the system is functioning and component 2 is working, what is the probability that component 1 is working?

Task 7 – Random Variables

Assume that we roll a fair 3-sided die three times. Here, the sides have values $1, 2, 3$.

a) Describe the PMF of the random variable X giving the sum of the first two rolls.

b) Give the expectation $\mathbb{E}[X]$.

c) Compute $\mathbb{P}(X > 3)$.

d) Let Y be the random variable describing the sum of the three rolls. Compute $\mathbb{P}(X = 5 \mid Y = 7)$.