CSE 312
Foundations of Computing II

Lecture 15: Exponential and Normal Distribution

Slide Credit: Based on Stefano Tessaro’s slides for 312 19au
incorporating ideas from Alex Tsun’s and Anna Karlin’s slides for 312 20su and 20au
Review – Continuous RVs

Probability Density Function (PDF).
\( f: \mathbb{R} \to \mathbb{R} \) s.t.
- \( f(x) \geq 0 \) for all \( x \in \mathbb{R} \)
- \( \int_{-\infty}^{+\infty} f(x) \, dx = 1 \)

Cumulative Density Function (CDF).
\[
F(y) = \int_{-\infty}^{y} f(x) \, dx
\]

Theorem. \( f(x) = \frac{dF(x)}{dx} \)

Density ≠ Probability!

\[
\mathbb{P}(X \in [a, b]) = \int_{a}^{b} f_X(x) \, dx = F_X(b) - F_X(a)
\]

\( F(y) = \mathbb{P}(X \leq y) \)
Expectation of a Continuous RV

**Definition.** The expected value of a continuous RV $X$ is defined as

$$
\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx
$$

**Fact.** $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

**Definition.** The variance of a continuous RV $X$ is defined as

$$
\text{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot (x - \mathbb{E}(X))^2 \, dx = \mathbb{E}(X^2) - \mathbb{E}(X)^2
$$
Expectation of a Continuous RV

Definition.

\[ \mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx \]

Example. \( T \sim \text{Unif}(0,1) \)

\[ f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases} \]

\[ f_T(x) \cdot x = \begin{cases} x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases} \]

\[ \mathbb{E}(T) = \frac{1}{2} \cdot 1^2 = \frac{1}{2} \]

Area of triangle
Uniform Distribution

\[ X \sim \text{Unif}(a, b) \]

We also say that \( X \) follows the uniform distribution / is uniformly distributed.

\[
f_X(x) = \begin{cases} 
\frac{1}{b-a} & x \in [a, b] \\
0 & \text{else} 
\end{cases}
\]
Uniform Density – Expectation

\[ X \sim \text{Unif}(a, b) \]

\[ \mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx \]

\[ = \frac{1}{b-a} \int_{a}^{b} x \, dx \]

\[ = \frac{1}{b-a} \left[ \frac{x^2}{2} \right]_{a}^{b} \]

\[ = \frac{1}{b-a} \left( \frac{b^2-a^2}{2} \right) \]

\[ = \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2} \]

\[ f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases} \]
Uniform Density – Variance

\[ X \sim \text{Unif}(a, b) \]

\[ \mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, dx \]

\[ = \frac{1}{b-a} \int_{a}^{b} x^2 \, dx = \frac{1}{b-a} \left( \frac{x^3}{3} \right) \bigg|_{a}^{b} = \frac{b^3 - a^3}{3(b-a)} \]

\[ = \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3} \]

\[ f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases} \]
Uniform Density – Variance

$X \sim \text{Unif}(a, b)$

$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$

$= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12}$

$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b - a)^2}{12}$

$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3}$

$\mathbb{E}(X) = \frac{a + b}{2}$
Uniform Distribution

We also say that $X$ follows the uniform distribution / is uniformly distributed.

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}(X) = \frac{a+b}{2}$$

$$\text{Var}(X) = \frac{(b-a)^2}{12}$$
Exponential Density

Assume expected # of occurrences of an event per unit of time is $\lambda$
- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER

Numbers of occurrences of event: Poisson distribution

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$  
(Discrete)

How long to wait until next event? Exponential density!

Let’s define it and then derive it!
The Exponential PDF/CDF

Assume expected # of occurrences of an event per unit of time is $\lambda$

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0,1,2,\ldots\}$
- Let $Y \sim \text{Exp}(\lambda)$ be the time till the first event. We will compute $F_Y(t)$ and $f_Y(t)$
- Let $X \sim \text{Poi}(t\lambda)$ be the # of events in the first $t$ units of time, for $t \geq 0$.
- $P(Y > t) = P(\text{no event in the first t units}) = P(X = 0) = e^{-t\lambda} \frac{t\lambda^0}{0!} = e^{-t\lambda}$
- $F_Y(t) = 1 - P(Y > t) = 1 - e^{-t\lambda}$
- $f_Y(t) = \frac{d}{dt} F_Y(t) = \lambda e^{-t\lambda}$
**Exponential Distribution**

**Definition.** An *exponential random variable* $X$ with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \text{Exp}(\lambda)$ and say $X$ that follows the exponential distribution.

**CDF:** For $y \geq 0$,

$$F_X(y) = 1 - e^{-\lambda y}$$

- $\lambda = 2$
- $\lambda = 1.5$
- $\lambda = 1$
- $\lambda = 0.5$
Expectation

\[
\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, dx
\]

\[
= \int_{0}^{+\infty} \lambda e^{-\lambda x} \cdot x \, dx
\]

\[
= \left. \left( -(x + \frac{1}{\lambda}) e^{-\lambda x} \right) \right|_{0}^{\infty} = \frac{1}{\lambda}
\]

Somewhat complex calculation use integral by parts

\[
f_X(x) = \begin{cases} 
\lambda e^{-\lambda x} & x \geq 0 \\
0 & x < 0
\end{cases}
\]

\[
\mathbb{E}(X) = \frac{1}{\lambda}
\]

\[
\text{Var}(X) = \frac{1}{\lambda^2}
\]
Memorylessness

**Definition.** A random variable is **memoryless** if for all $s, t > 0$,

$$
\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).
$$

**Fact.** $X \sim \text{Exp}(\lambda)$ is memoryless.

Assuming exp distr, if you’ve waited $s$ minutes, prob of waiting $t$ more is exactly same as $s = 0$.
Memorylessness of Exponential

**Fact.** \( X \sim \text{Exp}(\lambda) \) is memoryless.

**Proof.**

\[
\mathbb{P}(X > s + t \mid X > s) = \frac{\mathbb{P}(\{X > s + t\} \cap \{X > s\})}{\mathbb{P}(X > s)} = \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t)
\]

The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous).
example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins.
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins.

\[
T \sim \text{Exp}\left(\frac{1}{10}\right)
\]

\[
P\left(10 \leq T \leq 20\right) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} \, dx
\]

\[
y = \frac{x}{10}, \quad dy = \frac{dx}{10}
\]

\[
P\left(10 \leq T \leq 20\right) = \int_{1}^{2} e^{-y} \, dy = -e^{-y}\bigg|_{1}^{2} = e^{-1} - e^{-2}
\]
The Normal Distribution

Definition. A Gaussian (or normal) random variable with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

$$f_X(x) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

(We say that $X$ follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$, and $\text{Var}(X) = \sigma^2$

Proof is easy because density curve is symmetric around $\mu$, $f_X(\mu - x) = f_X(\mu + x)$

We will see next time why the normal distribution is (in some sense) the most important distribution.
The Normal Distribution

Aka a “Bell Curve” (imprecise name)