

CSE 312

Foundations of Computing II


Lecture 5: Intro to Discrete Probability



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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Agenda

- Events 
- Probability
- Equally Likely Outcomes
- Probability Axioms and Beyond Equally Likely Outcomes
- More Examples

Probability

- We want to model a non-deterministic process.
 - i.e., outcome not determined a-priori
 - E.g. throwing dice, flipping a coin...
 - We want to numerically measure likelihood of outcomes = probability.
 - We want to make complex statements about these likelihoods.
- We will not argue why a certain physical process realizes the probabilistic model we study
 - Why is the outcome of the coin flip really “random”?
- First part of class: “Discrete” probability theory
 - Experiment with finite / discrete set of outcomes.
 - Will explore countably infinite and continuous outcomes later

Sample Space

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Events

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Definition. Events E and F are **mutually exclusive** if $E \cap F = \emptyset$ (i.e., can't happen at same time)

Examples:

- For dice rolls: If $E = \{2, 4, 6\}$ and $F = \{1, 5\}$, then $E \cap F = \emptyset$

Example: 4-sided Dice

Suppose I roll two 4-sided dice Let D_1 be the value of the blue die and D_2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$

B. $D_1 + D_2 = 6$

C. $D_1 = 2 * D_2$

		Die 2 (D_2)			
		1	2	3	4
Die 1 (D_1)	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice

Suppose I roll two 4-sided dice Let D_1 be the value of the blue die and D_2 be the value of the red die. To the right is the sample space (possible outcomes).

What outcomes match these events?

A. $D_1 = 1$

$$A = \{(1,1), (1,2), (1,3), (1,4)\}$$

B. $D_1 + D_2 = 6$

$$B = \{(2,4), (3,3), (4,2)\}$$

C. $D_1 = 2 * D_2$

$$C = \{(2,1), (4,2)\}$$

Die 1 (D_1)

Die 2 (D_2)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Example: 4-sided Dice, Mutual Exclusivity

Are A and B mutually exclusive?

How about B and C ?

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	A & B	B & C
(a)	Yes	Yes
(b)	Yes	No
(c)	No	Yes
(d)	No	No

A. $D1 = 1$

$$A = \{(1,1), (1,2), (1,3), (1,4)\}$$

B. $D1 + D2 = 6$

$$B = \{(2,4), (3,3), (4,2)\}$$

C. $D1 = 2 * D2$

$$C = \{(2,1), (4,2)\}$$

Die 1 ($D1$)

Die 2 ($D2$)

	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

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Idea: Probability

A **probability** is a number (between 0 and 1) describing how likely a particular outcome will happen.

Will define a function

$$\mathbb{P}(\omega): \Omega \rightarrow [0, 1]$$

that maps outcomes $\omega \in \Omega$ to probabilities.

– Also use notation: $\mathbb{P}(\omega) = P(\omega) = \text{Pr}(\omega)$

Example – Coin Tossing

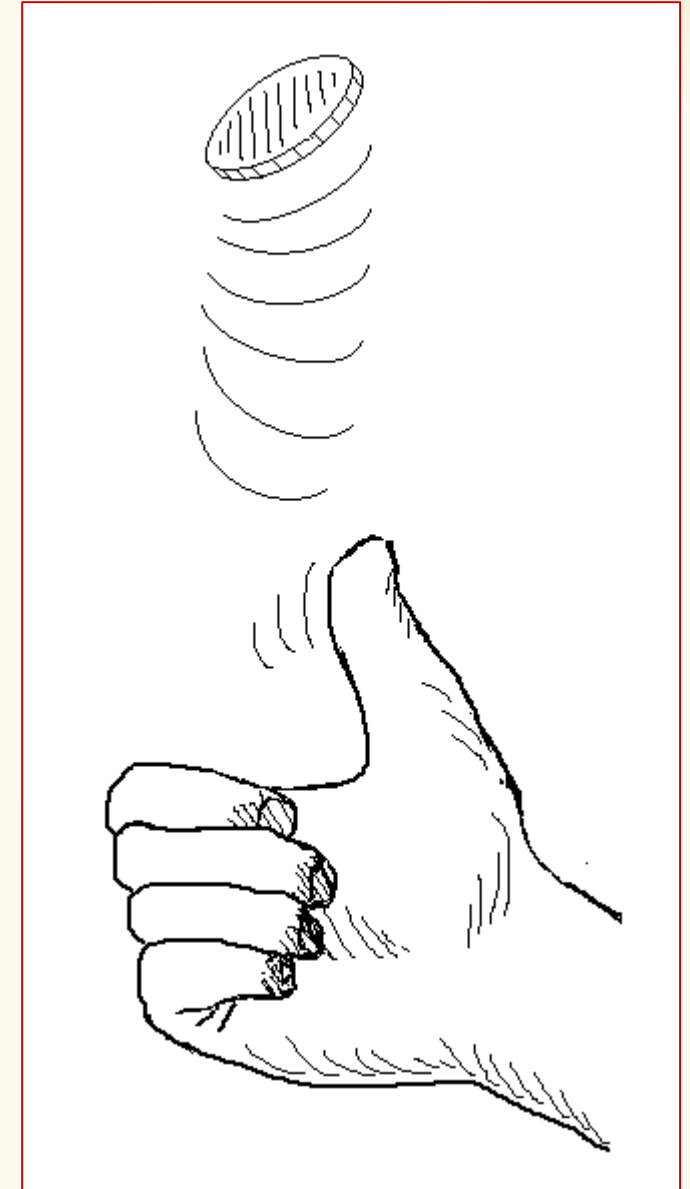
Imagine we toss one coin – outcome can be **heads** or **tails**.

$$\Omega = \{H, T\}$$

\mathbb{P} ? Depends! What do we want to model?!

Fair coin toss

$$\mathbb{P}(H) = \mathbb{P}(T) = \frac{1}{2} = 0.5$$



Example – Coin Tossing

Imagine we toss one coin – outcome can be **heads** or **tails**.

$$\Omega = \{H, T\}$$

\mathbb{P} ? Depends! What do we want to model?!

Bent coin toss (e.g., biased or unfair coin)

$$\mathbb{P}(H) = 0.85, \quad \mathbb{P}(T) = 0.15$$

Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space** is a pair (Ω, \mathbb{P}) where:

- Ω is a set called the **sample space**.

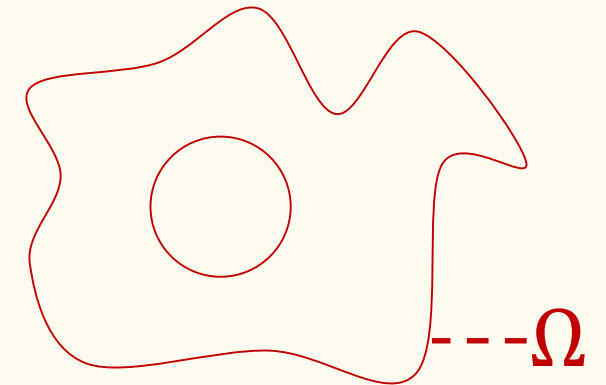
- \mathbb{P} is the **probability measure**,

a function $\mathbb{P}: \Omega \rightarrow [0,1]$ such that:

- $\mathbb{P}(x) \geq 0$ for all $x \in \Omega$

- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$

Set of possible elementary outcomes



Specify Likelihood (or probability) of each elementary outcome

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Uniform Probability Space

Definition. A uniform probability space is a pair (Ω, \mathbb{P}) such that

$$\mathbb{P}(x) = \frac{1}{|\Omega|}$$

for all $x \in \Omega$.

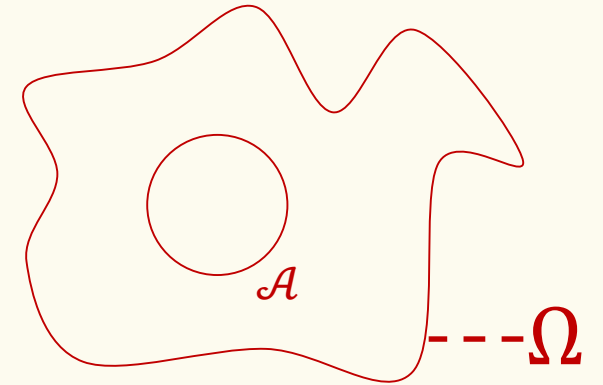
Examples:

- Fair coin $P(x) = \frac{1}{2}$
- Fair 6-sided die $P(x) = \frac{1}{6}$

Events

Definition. An **event** in a probability space (Ω, \mathbb{P}) is a subset $\mathcal{A} \subseteq \Omega$. Its probability is

$$\mathbb{P}(\mathcal{A}) = \sum_{\omega \in \mathcal{A}} \mathbb{P}(\omega)$$



Convenient abuse of notation: \mathbb{P} is extended to be defined over **sets**. $\mathbb{P}(\omega) = \mathbb{P}(\{\omega\})$

Care if the argument is an event or outcome!

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Example: 4-sided Dice, Event Probability

Think back to 4-sided die. Suppose each die is fair. What is the probability of event B ? $\Pr(B) = ???$

$$B. D1 + D2 = 6$$

$$B = \{(2,4), (3,3), (4,2)\}$$

		Die 2 (D2)			
		1	2	3	4
Die 1 (D1)	1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
	2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
	3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
	4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Equally Likely Outcomes

If (Ω, P) is a **uniform** probability space, then for any event $E \subseteq \Omega$, then

$$P(E) = \frac{|E|}{|\Omega|}$$

This follows from the definitions of the prob. of an event and uniform probability spaces.

Example – Coin Tossing

Toss a coin 100 times. Each outcome is **equally likely** (and assume the outcome of one toss does not impact another). What is the probability of seeing 50 heads?

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- (a) $\frac{1}{2}$
- (b) $\frac{1}{2^{50}}$
- (c) $\frac{\binom{100}{50}}{2^{100}}$
- (d) Not sure

Brain Break



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Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is more general to **any** probability space (not just uniform)

Axiom 1 (Non-negativity): $P(E) \geq 0$.

Axiom 2 (Normalization): $P(\Omega) = 1$

Axiom 3 (Countable Additivity): If E and F are mutually exclusive, then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$.

Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$

Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Review Probability space

Either finite or infinite countable (e.g., integers)

Definition. A (discrete) **probability space**

is a pair (Ω, \mathbb{P}) where:

- Ω is a set called the **sample space**.

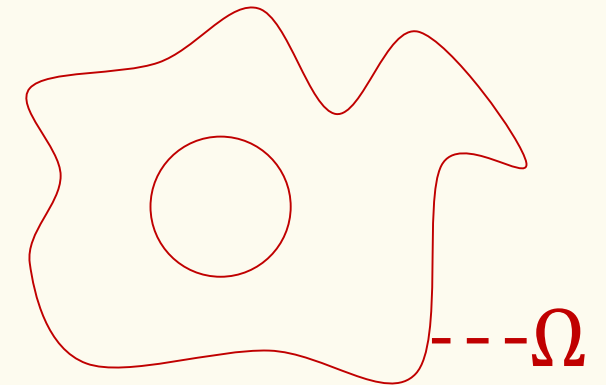
- \mathbb{P} is the **probability measure**,

a function $\mathbb{P}: \Omega \rightarrow \mathbb{R}$ such that:

- $\mathbb{P}(x) \geq 0$ for all $x \in \Omega$

- $\sum_{x \in \Omega} \mathbb{P}(x) = 1$

Set of possible elementary outcomes



Specify Likelihood (or probability) of each elementary outcome

Some outcome must show up

The likelihood (or probability) of each outcome is non-negative.

Non-equally Likely Outcomes

Many probability spaces can have **non-equally likely outcomes**.

Examples:

- Biased Coin: $P(H) = p$, $P(T) = 1 - P(H) = 1 - p$
- Glued coin: $P(HH) = P(TT) = 0$, $P(HT) = 0.5$, $P(TH) = 0.5$



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Example: Dice Rolls

Suppose I had a two, fair, 6-sided dice that we roll once each. What is the probability that we see *at least one 3 in the two rolls*.

Example: Birthday “Paradox”

Suppose we have a collection of n people in a room. What is the probability that at least 2 people share a birthday? Assuming there are 365 possible birthdays, with uniform probability for each day.

Example: Birthday “Paradox” cont.