Lecture 2: Permutation and Combinations
Last Class: Counting
• Binomial Coefficients, Binomial Theorem
• Inclusion-Exclusion
• Pigeon Hole Principle

Today: More Counting Practices
First Rule of Counting

**Product Rule:** In a sequential process, there are
- $n_1$ choices for the first step,
- $n_2$ choices for the second step (given the first choice), …, and
- $n_m$ choices for the $m^{th}$ step (given the previous choices),
then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$

**Application.** # of $k$-element sequences of distinct symbols
(a.k.a. $k$-permutations) from $n$-element set is

$$P(n, k) = n \times (n - 1) \times \cdots \times (n - k + 1) = \frac{n!}{(n - k)!}$$
Second Rule of Counting

**Combination:** If order does not matter, then count the number of ordered objects, and then divide by the number of orderings

**Applications.** The number of subsets of size $k$ of a set of size $n$ is

$$\binom{n}{k} = \frac{n!}{k! (n - k)!}$$

**Binomial coefficient** (verbalized as “$n$ choose $k$”)
Quick Review of Cards

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

How many possible 5 card hands?

\[ \binom{52}{5} \]
Counting Cards I

- A "straight" is five consecutive rank cards of any suit. How many possible straights?

\[ 10 \cdot 4^5 = 10,240 \]
Counting Cards II

- A flush is five card hand all of the same suit.
  How many possible flushes?

\[ 4 \cdot \binom{13}{5} = 5148 \]
Counting Cards III

- A flush is five card hand all of the same suit.
  How many possible flushes?

\[ 4 \cdot \binom{13}{5} = 5148 \]

- How many flushes are NOT straights?

\[ = \#\text{flush} - \#\text{flush and straight} \]

\[ (4 \cdot \binom{13}{5} = 5148) - 10 \cdot 4 \]

- 52 total cards
- 13 different ranks: 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A
- 4 different suits: Hearts, Diamonds, Clubs, Spades
Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards. \( \binom{4}{3} \cdot \binom{49}{2} \)

**Poll:**
A. correct  
B. Overcount  
C. Undercount
Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence $\square$ under counting Many sequences $\square$ over counting

**EXAMPLE:** How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards. \( \binom{4}{3} \cdot \binom{49}{2} \)

**Problem:** Many sequences lead to hands with 4 Aces

Eg. AC, AD, AH, AS, 2H
Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the unique sequence of choices that led to it.

No sequence under counting Many sequences over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

Use the sum rule

= # 5 card hand containing exactly 3 Aces
+ # 5 card hand containing exactly 4 Aces

\[
\binom{4}{3} \cdot \binom{48}{2} + \binom{48}{1}
\]
8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column?

Sequential process:
1. Column for pawn
2. Row for pawn
3. Column for bishop
4. Row for bishop
5. Column for knight
6. Row for knight

$(8 \cdot 7 \cdot 6)^2$
Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don’t share a row or a column

Pretend Rooks are different
1. Column for rook1
2. Row for rook1
3. Column for rook2
4. Row for rook2

Remove the order between two rooks

\[(8 \cdot 7)^2 / 2\]
Anagrams

How many ways can you arrange the letters in “Godoggy”?

\[ n = 7 \text{ Letters, } k = 4 \text{ Types } \{G, O, D, Y\} \]

\[ n_1 = 3, \ n_2 = 2, \ n_3 = 1, \ n_4 = 1 \]

\[
\frac{7!}{3! \ 2! \ 1! \ 1!} = \binom{7}{3,2,1,1}
\]

Multinomial coefficients
Multinomial Coefficients

If we have $k$ types of objects ($n$ total), with $n_1$ of the first type, $n_2$ of the second, ..., and $n_k$ of the $k^{th}$, then the number of ordering possible is

\[
\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!}
\]
The number of ways to distribute $n$ indistinguishable balls into $k$ distinguishable bins is

$$\binom{n + k - 1}{k - 1} = \binom{n + k - 1}{n}$$

E.g., = # of ways to add $k$ non-negative integers up to $n$
doughnuts

You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, plain.

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

1. Identify balls
2. Identify bins

\[
\binom{12 + 5 - 1}{5 - 1}
\]
You go to top pot to buy a dozen donuts. Your choices are Chocolate, Lemon-filled, Maple, Glazed, plain. How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?

Mental process:
1. Place one donut in each flavor bin
2. Choose the remaining 7 donuts without restriction

\[
\binom{7 + 5 - 1}{5 - 1}
\]
Inclusion-Exclusion

Let $A, B$ be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \text{singles} - \text{doubles} + \text{triples} - \text{quads} + \cdots$$

$$= (|A_1| + \cdots + |A_n|) - (|A_1 \cap A_2| + \cdots + |A_{n-1} \cap A_n|) + \cdots$$
Integers coprime with $N=\text{PQ}$

Let $N = P \times Q$ for two distinct prime numbers $P$ and $Q$. How many integers between 0 and $N - 1$ are coprime with $N$?

$a, N$ co-prime if no common divisor larger than 1

# integers between 0 and N-1
$- #$ integers between 0 and N-1 that share a non-trivial divisor with N
Integers coprime with $N = PQ$

$B = \{Q, 2Q, \ldots, PQ\}$

$|B| = P$

$A = \{P, 2P, \ldots, PQ\}$

$|A| = Q$

$A \cap B$ contains multiples of $P$ & $Q$

$A \cap B = \{0\}$

# Integers between 0 and $N-1$ that share a non-trivial divisor with $N$

$= |A| + |B| - |A \cap B| = P + Q - 1$

# Integers between 0 and $N-1$ that are co-prime with $N$

$= N - (P + Q - 1) = (p - 1) (q - 1)$
Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$
\textbf{Theorem.} Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

\textbf{Corollary.} For every even $n$, $0$ and $E$ are odd and even integers between $0$ and $n$

$$1 + \sum_{k \in 0} 2^k \binom{n}{k} = \sum_{k \in E} 2^k \binom{n}{k}$$

\textbf{Proof:} Set $x = 2, y = -1$ in the binomial theorem
Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Stars and bars
Counting is **NOT** for kindergarteners