CSE 312

Foundations of Computing II

Lecture 2: Permutation and Combinations

PAUL G. ALLEN SCHOOL
OF COMPUTER SCIENCE & ENGINEERING

Rachel Lin, Hunter Schafer

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Last Class: Counting

- Binomial Coefficients, Binomial Theorem
- Inclusion-Exclusion
- Pigeon Hole Principle

Today: More Counting Practices

First Rule of Counting

Product Rule: In a sequential process, there are

- n_1 choices for the first step,
- n_2 choices for the second step (given the first choice), … , and
- n_m choices for the m^{th} step (given the previous choices),

then the total number of outcomes is $n_1 \times n_2 \times \cdots \times n_m$

Application. # of k-element sequences of distinct symbols (a.k.a. k-permutations) from n-element set is

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Second Rule of Counting

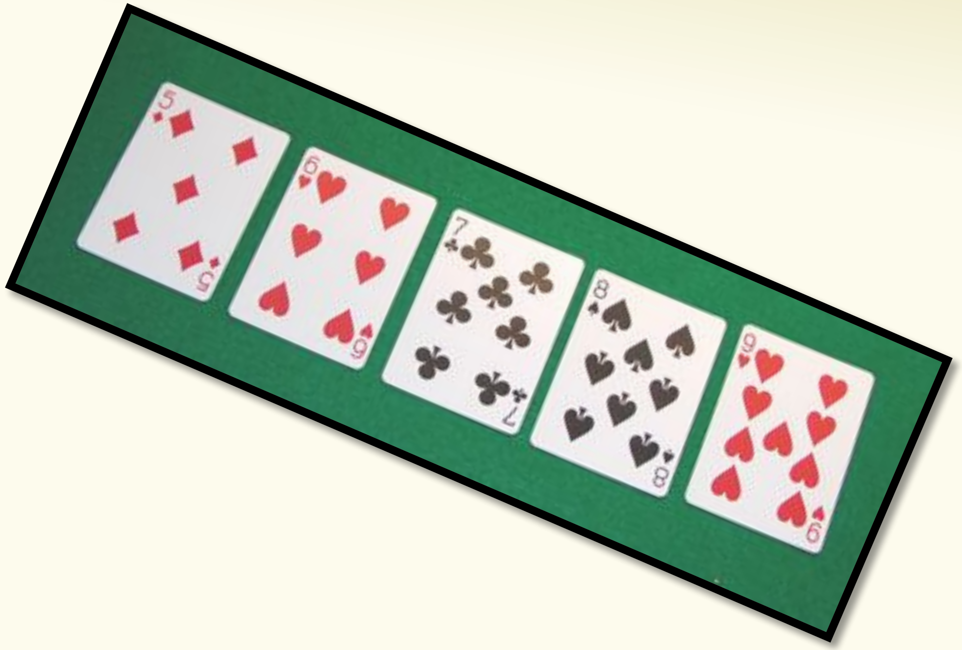
Combination: If order does not matter, then count the number of ordered objects, and then divide by the number of orderings

Applications. The number of subsets of size k of a set of size n is

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Binomial coefficient (verbalized as "n choose k")

Quick Review of Cards



How many possible 5 card hands?

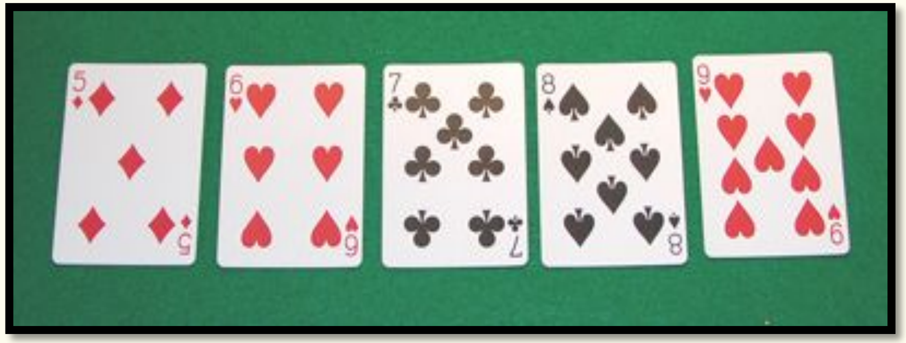
$$\binom{52}{5}$$

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

Counting Cards I

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A "straight" is five consecutive rank cards of any suit. How many possible straights?



$$10 \cdot 4^5 = 10,240$$

Counting Cards II

- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A flush is five card hand all of the same suit.

How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

Counting Cards III

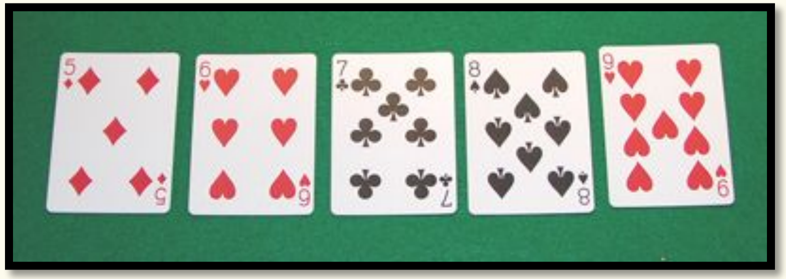
- 52 total cards
- 13 different ranks: 2,3,4,5,6,7,8,9,10,J,Q,K,A
- 4 different suits: Hearts, Diamonds, Clubs, Spades

- A flush is five card hand all of the same suit.

  How many possible flushes?

$$4 \cdot \binom{13}{5} = 5148$$

- How many flushes are NOT straights?



= #flush - #flush and straight

$$\left(4 \cdot \binom{13}{5} = 5148\right) - 10 \cdot 4$$

Sleuth’s Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence □ under counting

Many sequences □ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

First choose 3 Aces. Then choose remaining two cards.

$$\binom{4}{3} \cdot \binom{49}{2}$$

Poll:

A. correct

B. Overcount 

C. Undercount

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For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

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$$\binom{4}{3} \cdot \binom{49}{2}$$

Problem: Many sequences lead to hands with 4 Aces

Eg. AC, AD, AH, AS, 2H

Sleuth's Criterion (Rudich)

For each object constructed, it should be possible to reconstruct the **unique** sequence of choices that led to it.

No sequence □ under counting

Many sequences □ over counting

EXAMPLE: How many ways are there to choose a 5 card hand that contains at least 3 Aces?

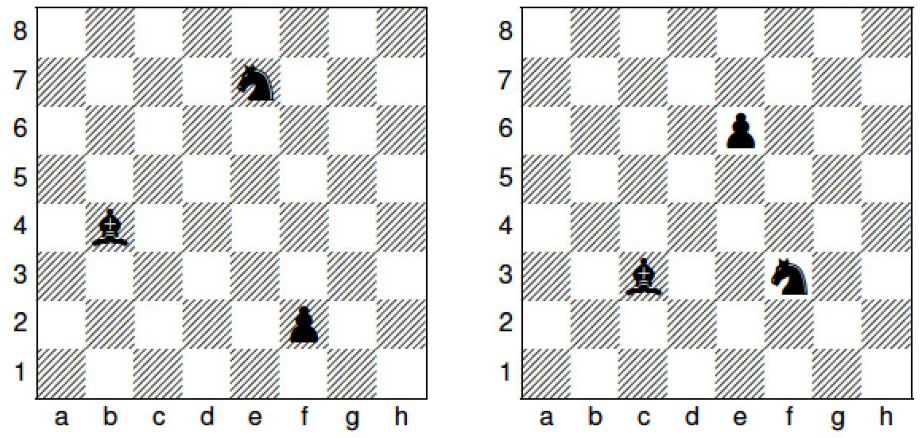
Use the sum rule

= # 5 card hand containing exactly 3 Aces — $\binom{4}{3} \cdot \binom{48}{2}$

+ # 5 card hand containing exactly 4 Aces — $\binom{48}{1}$

8 by 8 chessboard

How many ways to place a pawn, a bishop, and a knight so that none are in the same row or column ?



(a) valid (b) invalid

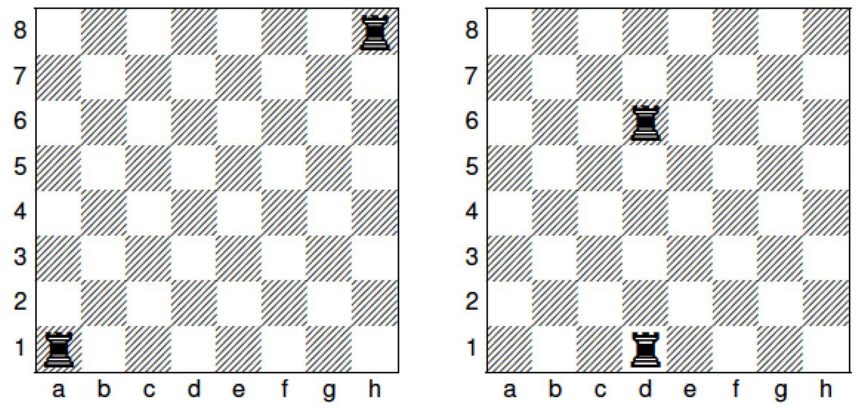
Sequential process:

1. Column for pawn
2. Row for pawn
3. Column for bishop
4. Row for bishop
5. Column for knight
6. Row for knight

$$(8 \cdot 7 \cdot 6)^2$$

Rooks on chessboard

How many ways to place two identical rooks on a chessboard so that they don't share a row or a column



(a) valid (b) invalid

Pretend Rooks are different

1. Column for rook1
2. Row for rook1
3. Column for rook2
4. Row for rook2

$(8 \cdot 7)^2$

Remove the order between two rooks

$(8 \cdot 7)^2/2$

Random Picture

Anagrams

How many ways can you arrange the letters in “Godoggy”?

$n = 7$ Letters, $k = 4$ Types {G, O, D, Y}

$n_1 = 3, n_2 = 2, n_3 = 1, n_4 = 1$

$$\frac{7!}{3!\,2!\,1!\,1!} = \binom{7}{3,2,1,1}$$

Multinomial coefficients

Multinomial Coefficients

If we have k types of objects ($\boldsymbol{n}$ total), with $\boldsymbol{n_1}$ of the first type, $\boldsymbol{n_2}$ of the second, … , and $\boldsymbol{n_k}$ of the k^{th}, then the number of ordering possible is

$$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{\mathrm{n!}}{n_1!\, n_2! \cdots n_k!}$$

Stars and Bars / Divider method

The number of ways to distribute n indistinguishable balls into k distinguishable bins is

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

E.g., = # of ways to add k non-negative integers up to n

doughnuts

You go to top pot to buy a dozen donuts. Your choices are

Chocolate, Lemon-filled, Maple, Glazed, plain

How many ways are there to choose a dozen doughnuts when doughnuts of the same type are indistinguishable?

1. Identify balls
2. Identify bins

$$\binom{12+5-1}{5-1}$$

doughnuts

You go to top pot to buy a dozen donuts. Your choices are

Chocolate, Lemon-filled, Maple, Glazed, plain

How many ways are there to choose a dozen doughnuts when you want at least 1 of each type?

Mental process:

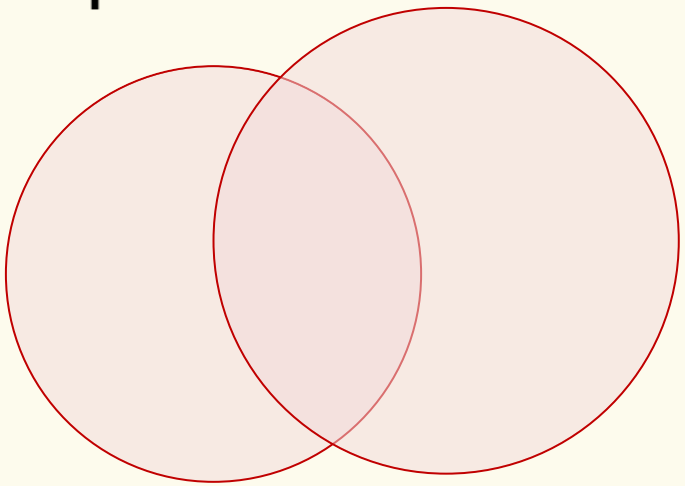
1. Place one donut in each flavor bin
2. Choose the remaining 7 donuts without restriction

$$\binom{7+5-1}{5-1}$$

Inclusion-Exclusion

Let A, B be sets. Then

$$|A \cup B| = |A| + |B| - |A \cap B|$$



In general, if $A_1, A_2, \ldots, A_n$ are sets, then

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = singles\ - doubles + triples\ - quads +\ \ldots$$
$$= (|A_1| + \cdots + |A_n|)\ - (|A_1 \cap A_2| +\ \ldots + |A_{n-1} \cap A_n|) +\ \ldots$$

Integers coprime with N=PQ

Let $N = P \times Q$ for two distinct prime numbers P and Q.
How many integers between 0 and $N - 1$ are coprime with N?

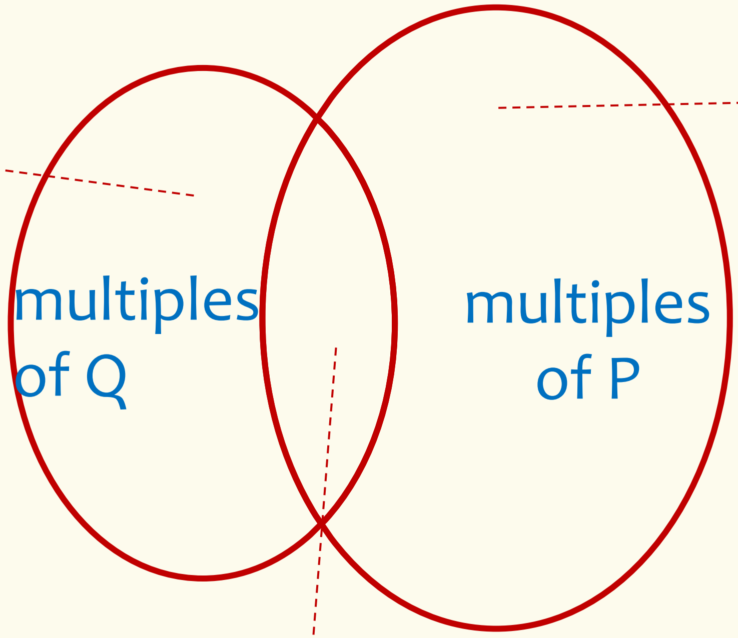
a, N co-prime if no common divisor larger than 1

integers between 0 and N-1
– # integers between 0 and N-1 that share a non-trivial divisor with N

Integers coprime with N=PQ

$B = \{Q, 2Q, \dots, PQ\}$

$|B| = P$



$A = \{P, 2P, \dots, PQ\}$

$|A| = Q$

$A \cap B$ contains multiples of P & Q $\quad A \cap B = \{0\}$

\# Integers between 0 and N-1 that share a non-trivial divisor with N
$= |A| + |B| - |A \cap B| = P + Q - 1$

\# Integers between 0 and N-1 that are co-prime with N
$= N - (P + Q - 1) = (p-1)(q-1)$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Binomial Theorem

Theorem. Let $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$ a positive integer. Then,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Corollary. For every **even** n, O and E are odd and even integers between 0 and n

$$1 + \sum_{k \in O} 2^k \binom{n}{k} = \sum_{k \in E} 2^k \binom{n}{k}$$

Proof: Set $x = 2, y = -1$ in the binomial theorem

Tools and concepts

- Sum rule, Product rule
- Permutations, combinations
- Inclusion-exclusion
- Binomial Theorem
- Combinatorial proofs
- Pigeonhole principle
- Stars and bars

Counting is NOT for kindergarteners