CSE 312

Foundations of Computing II

Lecture 9: Random Variables and Expectation

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incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Last Time

Theorem. (Chain Rule) For events $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n$,

$$\mathbb{P}(\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_n) = \mathbb{P}(\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_2 | \mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_3 | \mathcal{A}_1 \cap \mathcal{A}_2) \cdots \mathbb{P}(\mathcal{A}_n | \mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_{n-1})$$

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are (statistically) **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

"Equivalently." $\mathbb{P}(\mathcal{A} | \mathcal{B}) = \mathbb{P}(\mathcal{A})$.

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** conditioned on $\mathcal{C}$ if $\mathbb{P}(\mathcal{C}) \neq 0$ and $\mathbb{P}(\mathcal{A} \cap \mathcal{B} \mid \mathcal{C}) = \mathbb{P}(\mathcal{A} \mid \mathcal{C}) \cdot \mathbb{P}(\mathcal{B} \mid \mathcal{C})$.

Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

- *What is the total of two dice rolls?*
- *What is the number of coin tosses needed to see the first head?*
- *What is the number of heads among 2 coin tosses?*

Random Variables

Definition. A **random variable (RV)** for a probability space $(\Omega, \mathbb{P})$ is a function $X: \Omega \to \mathbb{R}$.

The set of values that X can take on is called its range/support $X(\Omega)$

Example. Number of heads in 2 independent coin flips $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$

RV Example

20 balls labeled 1, 2, … , 20 in a bin

- Draw a subset of 3 uniformly at random
- Let $X = \text{maximum of the 3 numbers on the balls}$
  - Example: $X(2, 7, 5) = 7$
  - Example: $X(15, 3, 8) = 15$

Poll: pollev.com/hunter312

A. 20^3
B. 20
C. 18
D. $\binom{20}{3}$

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Probability Mass Function (Idea)

Flipping two independent coins $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$

$X = \text{number of heads in the two flips}$

$X(HH) = 2 \qquad X(HT) = 1 \qquad X(TH) = 1 \qquad X(TT) = 0$

What is the support $X(\Omega)$? $X(\Omega) = \{0, 1, 2\}$

What is the probability that X is 2? To answer this, we introduce the notion of a **probability mass function (PMF)** that describes this probability.

$$Pr(X = k)$$

Probability Mass Function (PMF)

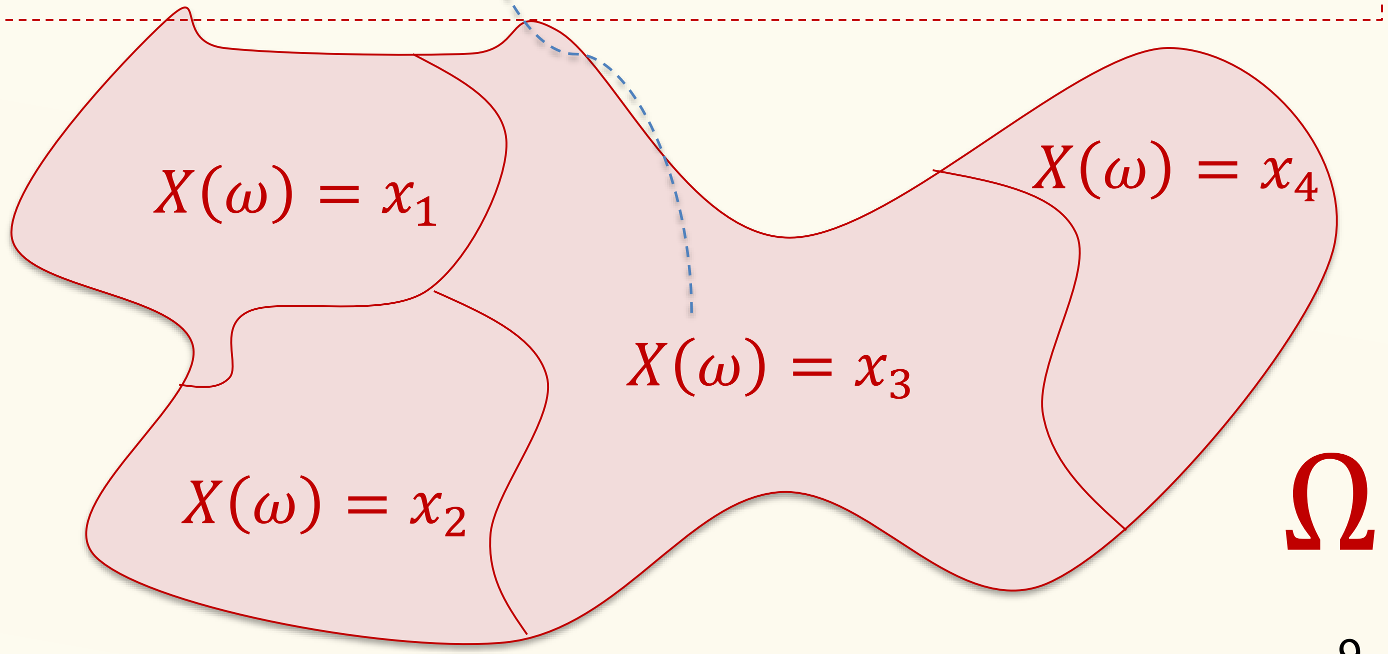
Definition. For a RV $X : \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} \stackrel{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $\mathbb{P}(X = x) = \mathbb{P}(\{X = x\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$ where $\mathbb{P}(X = x)$ is the probability mass function (PMF) of X

Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$



RV Example

20 balls labeled 1, 2, …, 20 in a bin

- Draw a subset of 3 uniformly at random
- Let $X = \text{maximum of the 3 numbers on the balls}$

What is $Pr(X = 20)$?

Poll: pollev.com/hunter312

A. $\binom{20}{2} / \binom{20}{3}$

B. $\binom{19}{2} / \binom{20}{3}$

C. $19^2 / \binom{20}{3}$

D. $19 \cdot 18 / \binom{20}{3}$


gifak-net
a
h
b
$S=\frac{a+b}{2}*h$

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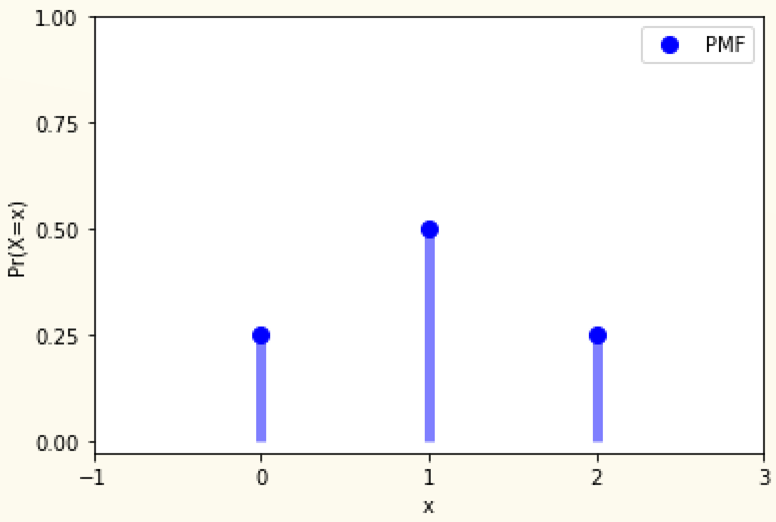
Cumulative Distribution Function (CDF)

Definition. For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of where X specifies for any real number x, the probability that $X \leq x$.

$$\mathrm{F}_X(x) = \Pr(X \leq x)$$

Go back to 2 coin clips, where X is the number of heads

$$\Pr(X = x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases} \qquad F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < 1 \\ \frac{3}{4}, & 1 \leq x < 2 \\ 1, & 2 \leq x \end{cases}$$



Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let X be the number of students who get their own HW

$\Pr(\omega)$	ω	$X(\omega)$
1/6	1, 2, 3	3
1/6	1, 3, 2	1
1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

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Expectation (Idea)

What is the *expected* number of heads in 2 independent flips of a fair coin?

Cumulative Disribution Function (CDF)

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the expectation or expected value of X is

$$\mathrm{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$$

or equivalently

$$\mathrm{E}[X] = \sum_{x \in X(\Omega)} x \cdot \Pr(X = x)$$

Intuition: “Weighted average” of the possible outcomes (weighted by probability)

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1/6	1, 2, 3	3
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1/6	2, 1, 3	1
1/6	2, 3, 1	0
1/6	3, 1, 2	0
1/6	3, 2, 1	1

Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability p of being heads. Keep flipping independent flips until heads. Let X be the number of flips until heads.

What is: $\Pr(X = 1) =$

What is: $\Pr(X = 2) =$

What is: $\Pr(X = k) =$

Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability p of being heads. Keep flipping independent flips until heads. Let X be the number of flips until heads. What is $\mathrm{E}[X]$?