Theorem. (Chain Rule) For events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$,
\[
\mathbb{P}(\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_n) = \mathbb{P}(\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_2 | \mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_3 | \mathcal{A}_1 \cap \mathcal{A}_2) \cdots \mathbb{P}(\mathcal{A}_n | \mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_{n-1})
\]

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are (statistically) independent if
\[
\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).
\]

“Equivalently.” $\mathbb{P}(\mathcal{A} | \mathcal{B}) = \mathbb{P}(\mathcal{A})$.

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are independent conditioned on $\mathcal{C}$ if
\[
\mathbb{P}(\mathcal{C} \neq 0) \quad \text{and} \quad \mathbb{P}(\mathcal{A} \cap \mathcal{B} | \mathcal{C}) = \mathbb{P}(\mathcal{A} | \mathcal{C}) \cdot \mathbb{P}(\mathcal{B} | \mathcal{C}).
\]
Agenda

• Random Variables
• Probability Mass Function (PMF)
• Cumulative Distribution Function (CDF)
• Expectation
Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

– *What is the total of two dice rolls?*
– *What is the number of coin tosses needed to see the first head?*
– *What is the number of heads among 2 coin tosses?*
Random Variables

**Definition.** A random variable (RV) for a probability space \((\Omega, \mathbb{P})\) is a function \(X: \Omega \rightarrow \mathbb{R}\).

The set of values that \(X\) can take on is called its range/support \(X(\Omega)\).

**Example.** Number of heads in 2 independent coin flips \(\Omega = \{HH, HT, TH, TT\}\)
RV Example

20 balls labeled 1, 2, ..., 20 in a bin

– Draw a subset of 3 uniformly at random
– Let $X = \text{maximum of the 3 numbers on the balls}$
  • Example: $X(2, 7, 5) = 7$
  • Example: $X(15, 3, 8) = 15$

Poll: pollev.com/hunter312

A. $20^3$
B. 20
C. 18
D. $\binom{20}{3}$
Agenda

• Random Variables
• Probability Mass Function (PMF)
• Cumulative Distribution Function (CDF)
• Expectation
Flipping two independent coins

\[ \Omega = \{HH, HT, TH, TT\} \]

\[ X = \text{number of heads in the two flips} \]

\[ X(HH) = 2 \quad X(HT) = 1 \quad X(TH) = 1 \quad X(TT) = 0 \]

What is the support \( X(\Omega) \)?

\[ X(\Omega) = \{0, 1, 2\} \]

What is the probability that \( X \) is 2? To answer this, we introduce the notion of a **probability mass function (PMF)** that describes this probability.

\[ Pr(X = k) \]
**Probability Mass Function (PMF)**

**Definition.** For a RV $X: \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} \overset{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $\mathbb{P}(X = x) = \mathbb{P}(\{X = x\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$ where $\mathbb{P}(X = x)$ is the probability mass function (PMF) of $X$.

Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$
RV Example

20 balls labeled 1, 2, ..., 20 in a bin
   – Draw a subset of 3 uniformly at random
   – Let $X = \text{maximum of the 3 numbers on the balls}$

What is $Pr(X = 20)$?

Poll: pollev.com/hunter312

A. $\frac{20}{3}$
B. $\frac{19}{3}$
C. $19^2$
D. $19 \cdot 18$
Agenda

• Random Variables
• Probability Mass Function (PMF)
• Cumulative Distribution Function (CDF)
• Expectation
**Cumulative Distribution Function (CDF)**

**Definition.** For a RV $X: \Omega \rightarrow \mathbb{R}$, the cumulative distribution function of where $X$ specifies for any real number $x$, the probability that $X \leq x$.

$$F_X(x) = \Pr(X \leq x)$$

Go back to 2 coin clips, where $X$ is the number of heads

$$\Pr(X = x) = \begin{cases} 
\frac{1}{4}, & x = 0 \\
\frac{1}{2}, & x = 1 \\
\frac{1}{4}, & x = 2 
\end{cases}$$

$$F_X(x) = \begin{cases} 
0, & x < 0 \\
\frac{1}{4}, & 0 \leq x < 1 \\
\frac{3}{4}, & 1 \leq x < 2 \\
1, & 2 \leq x
\end{cases}$$
Example: Returning Homeworks

• Class with 3 students, randomly hand back homeworks. All permutations equally likely.
• Let $X$ be the number of students who get their own HW

<table>
<thead>
<tr>
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Agenda

• Random Variables
• Probability Mass Function (PMF)
• Cumulative Distribution Function (CDF)
• Expectation
Expectation (Idea)

What is the *expected* number of heads in 2 independent flips of a fair coin?
Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the expectation or expected value of $X$ is

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot Pr(\omega)$$

or equivalently

$$E[X] = \sum_{x \in X(\Omega)} x \cdot Pr(X = x)$$

Intuition: “Weighted average” of the possible outcomes (weighted by probability)
Example: Returning Homeworks

• Class with 3 students, randomly hand back homeworks. All permutations equally likely.
• Let $X$ be the number of students who get their own HW

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Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability $p$ of being heads. Keep flipping independent flips until heads. Let $X$ be the number of flips until heads.

What is: $\text{Pr}(X = 1) =$

What is: $\text{Pr}(X = 2) =$

What is: $\text{Pr}(X = k) =$
Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability $p$ of being heads. Keep flipping independent flips until heads. Let $X$ be the number of flips until heads. What is $E[X]$?