Agenda

• Recap: Markov Chains
  – Intuition
  – Computing Probabilities
  – Matrix Notation
• Stationary Distributions
• PageRank
So far, a single-shot random process

Random Process $\rightarrow$ Outcome Distribution $D$

Many-step random process

Random Process 1 $\rightarrow$ Outcome Distribution $D_1$ $\rightarrow$ Random Process 2 $\rightarrow$ Outcome Distribution $D_2$ $\rightarrow$ Random Process 3 $\rightarrow$ Outcome Distribution $D_3$ $\rightarrow$ ...

**Definition:** A **discrete-time stochastic process** (DTSP) is a sequence of random variables $X^{(0)}, X^{(1)}, X^{(2)}, \ldots$ where $X^{(t)}$ is the value at time $t$.

Last time / Today:
See a very special type of DTSP called **Markov Chains**
Formalizing Markov Chain

Markov Property

\[ P(x^{(t+1)} = j \mid x^{(t)} = i, \text{history}) = P(r_{x^{(t+1)} = j \mid x^{(t)} = i}) \]

\( X(t) \) my state at t \quad \( X(t+1) \) my state at t+1

3. What is the prob that I work at \( t+1 = 100 \)?

By LTP:

\[ p_{(W)}^{(t+1)} = P(X^{(t+1)} = W) = \sum_{U \in \{W, S, E\}} P(X^{(t+1)} = W \mid X^{(t)} = U) P(X^{(t)} = U) \]
Transition Matrix

$$P = \begin{pmatrix} .4 & .6 & 0 \\ .1 & .6 & .3 \\ .5 & 0 & .5 \end{pmatrix}$$

$$X(t) = (p_W(t), p_S(t), p_E(t))$$

LTP: $$p_W^{(t+1)} = P(X^{(t+1)} = W) = \sum_{U \in \{W,S,E\}} P(X^{(t+1)} = W \mid X(t) = U) \cdot P(X(t) = U)$$

$$\Rightarrow X^{(t+1)} = X(t) \cdot P$$

$$\Rightarrow X^{(t)} = X^{(0)} \cdot P^t$$

3. What is the prob that I work at $$t = 100$$?

Closed formula: $$p_W^{(t)} = X^{(t)}[1] = (X^{(0)} \cdot P^t)[1]$$
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Example: Random Walks

Suppose we start at node 1, and at each step transition to a neighboring node with equal probability.

How does the probability of me being at each node look as we let this process go on?

\[
\begin{align*}
Pr(X^{(t+1)} = 2 \mid X^{(t)} = 1) &= \frac{1}{2} \\
Pr(X^{(t+1)} = 3 \mid X^{(t)} = 1) &= \frac{1}{2}
\end{align*}
\]
Example: Random Walks

Start by defining transition probs.

\[ P_{ij} = P[i, j] = \Pr(X^{(t+1)} = j \mid X^{(t)} = i) \]

\[ p_i^{(t)} = \Pr(X^{(t)} = i) = \] (Equation)

\[ = X^{(t)} \mathbb{1}_i = (X^{(0)} P^t) \mathbb{1}_i \]
Example: Random Walks

Start by defining transition probs.

\[
p_{ij} = P[i, j] = \text{Pr}(X^{(t+1)} = j | X^{(t)} = i)
\]

\[
p_i^{(t)} = \text{Pr}(X^{(t)} = i) = \mathbb{E}[X^{(t)}] = (X^0 p^t)[i]
\]
Example: Random Walks

Compute $\Pr(X^{(2)} = 3 \mid X^{(0)} = 2) = \sum_{i=1}^{5} \Pr(X^{(2)} = 3 \mid X^{(1)} = i, X^{(0)} = 2) \Pr(X^{(1)} = i) = \sum_{i=1}^{5} \Pr(X^{(1)} = i) \Pr(X^{(0)} = 2 \mid X^{(1)} = i)$

$= \Pr(X^{(1)} = 3 \mid X^{(0)} = 1) \Pr(X^{(0)} = 1) + \Pr(X^{(1)} = 3 \mid X^{(0)} = 4) \Pr(X^{(0)} = 4)$

$= p_{13} \Pr(X^{(0)} = 1) + p_{43} \Pr(X^{(0)} = 4)$

$= p_{13} p_{21} + p_{43} p_{24}$

$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{12}$

$\Pr(X^{(0)} = 3) = \underbrace{p_{13} p_{21}}_{p_{21}} \Pr(X^{(0)} = 2)$
Example: Random Walks

Compute $\Pr(X^{(2)} = 3 \mid X^{(0)} = 2)$

$x^{(0)} = (0, 1, 0, 0, 0)$

$P = \begin{pmatrix}
0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}$

$x^{(1)} = \left( 0, 1, 0, 0, 0 \right)$

$x^{(2)} = x^{(1)} P = \left( \frac{1}{2}, 0, 0, \frac{1}{2}, 0 \right) \cdot \left( \cdots P \cdots \right)$

$= \left( 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2} \right)$

$= \left( 0, \frac{5}{12}, \frac{5}{12}, 0, \frac{1}{6} \right)$

$\Pr(X^{(2)} = 3 \mid X^{(0)} = 2)$
**Definition.** The **stationary distribution of a Markov Chain** with \( n \) states (which doesn’t always exist), is the \( n \)-dimensional row vector \( \pi \) (which must be a probability distribution – nonnegative and sums to 1) such that

\[
\pi P = \pi
\]

Intuition: Distribution over states at next step is the same as the distribution over states at the current step.
Intuition: \( x(t) \) is the distribution of being at each state at time \( t \) computed by \( x(t) = x(0) P^t \). As \( t \) is large \( x(t) \approx x^{(st)} \).

Theorem. The **Fundamental Theorem of Markov Chains** says that (under some minor technical conditions), for a Markov Chain with transition probabilities \( P \) and for any starting distribution over the states \( x^{(0)} \)

\[
\lim_{t \to \infty} x^{(0)} P^t = \pi
\]

where \( \pi \) is the stationary distribution of \( P \) (i.e., \( \pi P = \pi \))
Brain Break
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PageRank: Some History

The year was 1997
- Bill Clinton in the White House
- Deep Blue beat world chess champion (Kasparov)

The internet was not like it was today. Finding stuff was hard!
- In Nov 1997, only one of the top 4 search engines actually found itself when you searched for it
The Problem

Search engines worked by matching words in your queries to documents.

Not bad in theory, but in practice there are lots of documents that match a query.

– Search for Bill Clinton, top result is ‘Bill Clinton Joke of the Day’
– Susceptible to spammers and advertisers
The Fix: Ranking Results

Start by doing filtering to relevant documents (that part is easier). Then **rank** the results based on some measure of ‘quality’ or ‘authority’.

Key question: Who defines ‘quality’ or ‘authority’?

Enter two groups:
- Jon Kleinberg (professor at Cornell, MacArthur Genius Prize)
- Larry Page and Sergey Brin (Ph.D. students at Stanford, founded Google)
PageRank - Idea

Use **hyperlink analysis** to compute what pages are high quality or have high authority. Trust the internet itself define what is useful.

Define $q$ to be quality vector s.t. $q_A =$ quality of $A$

$$q_A = \frac{1}{2} q_G + 1 q_D + \frac{1}{2} q_F + 1 q_B$$

$$q_H = \frac{1}{3} q_A$$

Define $P_{ij} = \begin{cases} \text{Outdegree(i)} & \text{if } i \to j \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow$ Find $q$ s.t. $qP = q$. 
PageRank

Idea: Use this transition matrix $P$ to compute quality of webpages. Namely, find $q$ such that

$$qP = q$$

Seems like trying to find the stationary distribution of a Markov chain? Where is the Markov chain here? A random surfer!

– Starts at some node (webpage) and randomly follows a link to another.
– Use stationary distribution of her surfing patterns after a long time as notion of quality
Issues with PageRank

• How to handle dangling nodes (dead ends)?
• How to handle Rank sinks – group of pages that only link to each other?

Both solutions can be solved by “teleportation”
Final PageRank Algorithm

• Make a Markov Chain with one state for each webpage on the internet with the transition probabilities $P_{ij} = \frac{1}{\text{outdeg}(i)}$.

• Use a modified random walk. At each point in time, the surfer is at some webpage $x$.
  – With probability $p$, take a step to one of the neighbors of $x$ (equally likely)
  – With probability $1 - p$, “teleport” to a uniformly random page in the whole internet.

• Compute stationary distribution $\pi$ of this perturbed Markov chain.

• Define the PageRank of a webpage $x$ as the stationary probability $\pi_x$.

• Order pages by PageRank
PageRank - Example
It Gets More Complicated

While this basic algorithm was the defining thing that launched Google on their path to success, this is not the end to optimizing search.

Nowadays, Google has a LOT more secret sauce to ranking pages most of which they don’t reveal for 1) competitive advantage and 2) avoid gaming their algorithm.