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CSE 312

Foundations of Computing II

Lecture 22: Tail Bounds

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Music: Peach Pit

Joint PMFs and Joint Range

Definition. Let X and Y be discrete random variables. The **Joint PMF** of X and Y is

$$p_{X,Y}(a,b) = \Pr(X = a, Y = b)$$

Definition. The **joint range** of $p_{X,Y}$ is

$$\Omega(X,Y) = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega(X) \times \Omega(Y)$$

Note that

$$\sum_{(s,t)\in\Omega(X,Y)} p_{X,Y}(s,t) = 1$$

Law of Total Expectation

Law of Total Expectation (event version). Let X be a random variable and let events $A_1, \dots, A_n$ partition the sample space. Then,

$$E[X] = \sum_{i=1}^{n} E[X|A_i]\Pr(A_i)$$

Law of Total Expectation (random variable version). Let X be a random variable and Y be a discrete random variable. Then,

$$E[X] = \sum_{y \in \Omega(Y)} E[X|Y = y]\Pr(Y = y)$$

Example: Computer Failures

Suppose your computer operates in a sequence of steps, and that at each step i your computer will fail with probability p (independently of other steps). Let X be the number of steps it takes your computer to fail. What is $E[X]$?

X = # steps until first failure

A = computer fails on step 1

$$E[X] = \underbrace{E[X \mid A]}\,\underbrace{\Pr(A)} + \underbrace{E[X \mid \bar{A}]}\,\underbrace{\Pr(\bar{A})}$$

$$= 1 \cdot p + (1 + E[X]) \cdot (1-p)$$

$$= 1 + (1-p)E[X]$$

$$\Rightarrow E[X] = 1/p$$

Fails on first step

$E[X \mid A] = 1$

$E[X \mid \bar{A}] = 1 + E[X]$

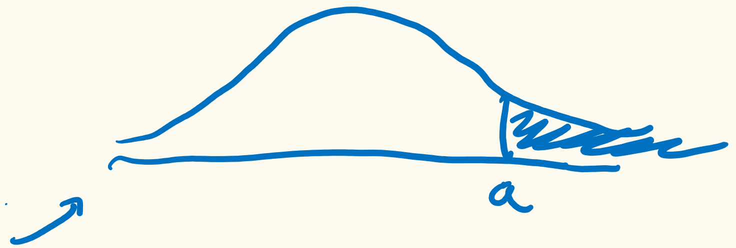
X is memorylessness

Agenda

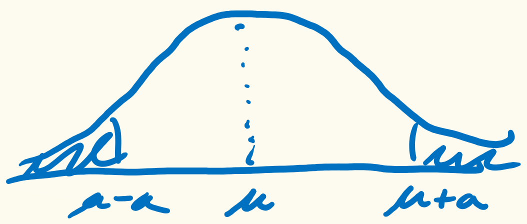
- Markov's Inequality
- Chebyshev's Inequality

Tail Bounds (Idea)

Bounding the probability a random variable is far from its mean. Usually statements of the form:



$$\Pr(X \geq a) \leq b$$
$$\Pr(|X - E[X]| \geq a) \leq b$$



Useful tool when

- An approximation that is easy to compute is sufficient
- The process is too complex to analyze exactly

Markov's Inequality

Example

$X \geq 0, \quad E[X] = 4$

$Pr(X \geq 12) \leq \frac{E[X]}{12} = \frac{4}{12} = \frac{1}{3}$

Theorem. Let X be a random variable taking only non-negative values. Then, for any $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

Incredibly simplistic – only requires that the random variable is non-negative and only needs you to know expectation. You don't need to know **anything else** about the distribution of X.

Markov's Inequality – Proof

Theorem. Let X be a (discrete) random variable taking only non-negative values. Then, for any $t > 0$,

$$\mathbb{P}(X \geq t) \leq \frac{\mathbb{E}(X)}{t}.$$

$$\mathbb{E}(X) = \sum_{x} x \cdot \mathbb{P}(X = x)$$

$$= \sum_{x \geq t} x \cdot \mathbb{P}(X = x) + \sum_{x < t} x \cdot \mathbb{P}(X = x)$$

≥ 0 because $x \geq 0$ whenever $\mathbb{P}(X = x) \geq 0$ (takes only non-negative values)

$$\geq \sum_{x \geq t} x \cdot \mathbb{P}(X = x)$$

$$\geq \sum_{x \geq t} t \cdot \mathbb{P}(X = x) \quad = t \cdot \mathbb{P}(X \geq t)$$

Follows by re-arranging terms …

$$\Rightarrow \Pr(X \geq t) \leq \frac{E[X]}{t}$$

Example – Geometric Random Variable

Let X be geometric RV with parameter p

$$\mathbb{P}(X = i) = (1-p)^{i-1}p \qquad \mathbb{E}(X) = \frac{1}{p}$$

"How many times does Alice need to flip a biased coin until she sees heads, if heads occurs with probability p?

What is the probability that $X \geq 2\mathbb{E}(X) = 2/p$?

Markov's inequality: $\mathbb{P}(X \geq 2/p) \leq \frac{\mathbb{E}(X)}{2/p} = \underset{\uparrow \atop E[X]}{\frac{1}{p}} \cdot \underset{\uparrow \atop 1/2/p}{\frac{p}{2}} = \frac{1}{2}$

Can we do better?

Example

$X \geq 0$, $E[X] = 25$

Suppose that the average number of ads you will see on a website is 25. Give an upper bound on the probability of seeing a website with 75 or more ads.

$$\Pr(X \geq 75) \leq \frac{E[X]}{75}$$

$$= \frac{25}{75}$$

$$= \frac{1}{3}$$

Poll: pollev.com/hunter312

a. $0 \leq p < 0.25$

b. $0.25 \leq p < 0.5$

c. $0.5 \leq p < 0.75$

d. $0.75 \leq p$

e. Unable to compute

Example

$X \geq 0, \quad E[X] = 25$

Suppose that the average number of ads you will see on a website is 25. Give an upper bound on the probability of seeing a website with 20 or more ads.

$$\Pr(X \geq 20) \leq \frac{E[X]}{20} = \frac{25}{20} = 1.25$$

Poll: pollev.com/hunter312

a. $0 \leq p < 0.25$
b. $0.25 \leq p < 0.5$
c. $0.5 \leq p < 0.75$
d. $0.75 \leq p$
~~e.~~ Unable to compute

Prob ≤ 1.25 is always true!

Brain Break

Agenda

- Markov's Inequality
- Chebyshev's Inequality

Chebyshev's Inequality

Markov: $Pr(X \geq t) \leq E[X]/t$

Theorem. Let X be a random variable. Then, for any $t > 0$,

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq t) \leq \frac{\mathrm{Var}(X)}{t^2}.$$

1) Need $E[X]$, $Var(X)$

2) X can be negative

Proof: Define $Z = X - \mathbb{E}(X)$

$E[Z^2] = E[(X-E[X])^2] = Var(X)$

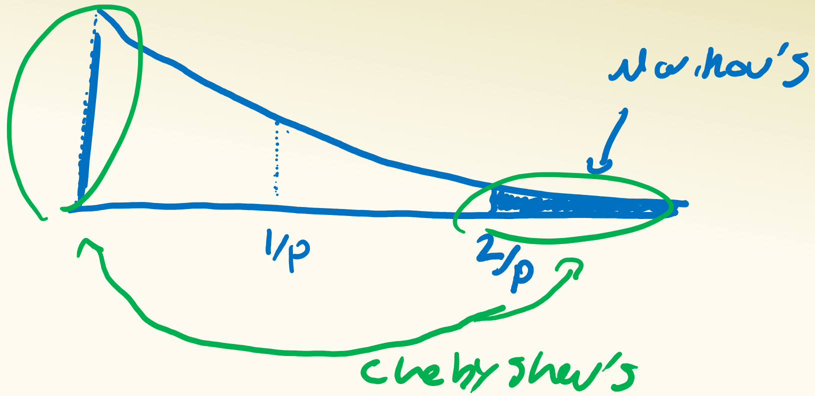
Definition of Variance

$$\mathbb{P}(|Z| \geq t) = \mathbb{P}(Z^2 \geq t^2) \leq \frac{\mathbb{E}(Z^2)}{t^2} = \frac{\mathrm{Var}(X)}{t^2}$$

$|Z| \geq t$ iff $Z^2 \geq t^2$

Markov's inequality ($Z^2 \geq 0$)

Example – Geometric Random Variable



Let X be geometric RV *with* parameter p

$$\mathbb{P}(X = i) = (1-p)^{i-1}p \qquad \mathbb{E}(X) = \frac{1}{p} \qquad \mathrm{Var}(X) = \frac{1-p}{p^2}$$

What is the probability that $X \geq 2\mathbb{E}(X) = 2/p$*?*

<u>Markov:</u> $\mathbb{P}(X \geq 2/p) \leq \frac{\mathbb{E}(X)}{2/p} = \frac{1}{p} \cdot \frac{p}{2} = \frac{1}{2}$

<u>Chebyshev:</u> $\mathbb{P}(X \geq 2/p) \leq \mathbb{P}\left(\left|X - \frac{1}{p}\right| \geq \frac{1}{p}\right) \leq \frac{\mathrm{Var}(X)}{1/p^2} = 1-p$

Not better, unless $p > 1/2$ ☹

Example

$Pr(|X - E[X]| \geq t) \leq Var(X)/t^2$

$\sigma = 5$

Use Chebyshev's

$Var(X) = \sigma^2 = 25$

Suppose that the average number of ads you will see on a website is 25 and the standard deviation of the number of ads is 5. Give an upper bound on the probability of seeing a website with 30 or more ads.

$Pr(X \geq 30) \leq Pr(|X - 25| \geq 5)$

$\leq Var(X)/5^2$

$= 25/25 = 1$

Poll: pollev.com/hunter312

a. $0 \leq p < 0.25$
b. $0.25 \leq p < 0.5$
c. $0.5 \leq p < 0.75$
d. $0.75 \leq p$
e. Unable to compute

Chebyshev's Inequality – Repeated Experiments

"How many times does Alice need to flip a biased coin <u>until she sees heads n</u> times, if heads occurs with probability p?

X = # of flips until n times "heads"

X_i = # of flips between $(i-1)$-st and i-th "heads"

$$X = \sum_i X_i$$

Note: $X_1, \dots, X_n$ are independent and geometric with parameter p

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_i X_i\right) = \sum_i \mathbb{E}(X_i) = \frac{n}{p}$$

$$\mathrm{Var}(X) = \sum_i \mathrm{Var}(X_i) = \frac{n(1-p)}{p^2}$$

Chebyshev's Inequality – Coin Flips

"How many times does Alice need to flip a biased coin <u>until she sees heads n</u> times, if heads occurs with probability p?

$$\mathbb{E}(X) = \mathbb{E}\left(\sum_i X_i\right) = \sum_i \mathbb{E}(X_i) = \frac{n}{p} \qquad \mathrm{Var}(X) = \sum_i \mathrm{Var}(X_i) = \frac{n(1-p)}{p^2}$$

What is the probability that $X \geq 2\mathbb{E}(X) = 2n/p$?

<u>Markov:</u> $\mathbb{P}(X \geq 2n/p) \leq \frac{\mathbb{E}(X)}{2n/p} = \frac{n}{p} \cdot \frac{p}{2n} = \frac{1}{2}$

<u>Chebyshev:</u> $\mathbb{P}(X \geq 2n/p) \leq \mathbb{P}\left(\left|X - \frac{n}{p}\right| \geq \frac{n}{p}\right) \leq \frac{\mathrm{Var}(X)}{n^2/p^2} = \frac{1-p}{n}$

Goes to zero as $n \to \infty$ ☺

Tail Bounds

Useful for approximations of complex systems. How good the approximation is depends on the actual distribution and the context you are using it in.

– Usually loose upper-bounds are okay when designing for worst-case

Generally (but not always) making more assumptions about your random variable leads to a more accurate upper-bounder.