Welcome! Ask Qs or say hi before/during/after class in chat!

CSE 312

Foundations of Computing II

Lecture 21: Joint Distributions

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Music: Sammy Rae & The Friends

Agenda

- Joint Distributions
  - Cartesian Products
  - Joint PMFs and Joint Range
  - Marginal Distribution
- Conditional Expectation and Law of Total Expectation

Review Cartesian Product

Definition. Let A and B be sets. The **Cartesian product** of A and B is denoted

$$A \times B = \{(a, b) : a \in A, b \in B\}$$

Example.

$$\{1,2,3\} \times \{4, 5\} = \{(1, 4), (1,5), (2,4), (2,5), (3, 4), (3,5)\}$$

If A and B are finite sets, then $|A \times B| = |A| \cdot |B|$.

The sets don't need to be finite! You can have $\mathbb{R} \times \mathbb{R}$ (often denoted $\mathbb{R}^2$)

Joint PMFs and Joint Range X, Y $P(X = \underline{a}, Y = \underline{b})$

Definition. Let X and Y be discrete random variables. The **Joint PMF** of X and Y is

if X, Y independent

$$\underline{p_{X,Y}(a,b)} = \Pr(X = a, Y = b) \overset{\downarrow}{=} Pr(X=a)Pr(Y=b)$$

Definition. The **joint range** of $p_{X,Y}$ is

$$\Omega(X,Y) = \{(c,d) : p_{X,Y}(c,d) > 0\} \subseteq \Omega(X) \times \Omega(Y)$$

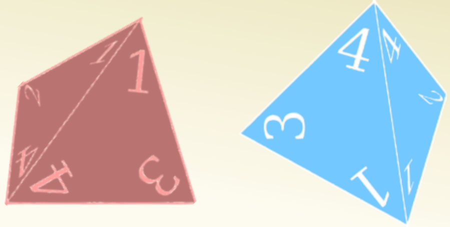
Note that

$$\sum_{(s,t)\in\Omega(X,Y)} p_{X,Y}(s,t) = 1$$

Example: Weird Dice

$p_{XY}(a,b) = Pr(X=a, Y=b)$

$p_{XY}(1,3) = Pr(X=1, Y=3)$



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die.

$P(X=1, Y=3)$
$= P(X=1)P(Y=3)$
$= (1/4)^2 = 1/16$

$\Omega(X) = \{1,2,3,4\}$ and $\Omega(Y) = \{1,2,3,4\}$

In this problem, the joint PMF is

$$p_{X,Y}(x,y) = \begin{cases} 1/16, & x, y \in \Omega(X,Y) \\ 0, & \text{otherwise} \end{cases}$$

X\Y	1	2	3	4
1	1/16	1/16	1/16	1/16
2	1/16	1/16	1/16	1/16
3	1/16	1/16	1/16	1/16
4	1/16	1/16	1/16	1/16

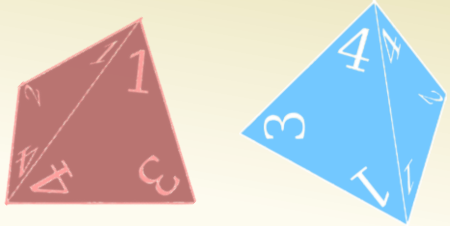
and the joint range is (since all combinations have non-zero probability)

$\Omega(X,Y) = \Omega(X) \times \Omega(Y)$ $= \{(1,1), (1,2), \ldots, (4,3), (4,4)\}$

Example: Weirder Dice

$p_{UW}(1,3) = \Pr(U=1, W=3)$

$p_{UW}(3,1) = \Pr(\min=3, \max=1) = 0$



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega(U) = \{1,2,3,4\}$ and $\Omega(W) = \{1,2,3,4\}$

In general:
$p_{UW}(a,b) \neq p_{UW}(b,a)$

$\Omega(U, W) = \{(u, w) \in \Omega(U) \times \Omega(W) : u \leq w\} \neq \Omega(U) \times \Omega(W)$

Poll: pollev.com/hunter312

What is $p_{U,W}(1,3) = \Pr(U = 1, W = 3)$?

a. 1/16
b. 2/16
c. 1/2
d. Not sure

$= \Pr(X=1, Y=3 \cup X=3, Y=1)$

$= \Pr(X=1, Y=3) + \Pr(X=3, Y=1)$

$= \Pr(X=1)\Pr(Y=3) + \Pr(X=3)\Pr(Y=1)$

$= 1/16 + 1/16 = 2/16$

U\W	1	2	3	4
1	1/16	2/16	$\boxed{2/16}$	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Example: Weirder Dice

Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

$\Omega(U) = \{1,2,3,4\}$ and $\Omega(W) = \{1,2,3,4\}$

$$\Omega(U, W) = \{(u, w) \in \Omega(U) \times \Omega(W) : u \leq w\} \neq \Omega(U) \times \Omega(W)$$

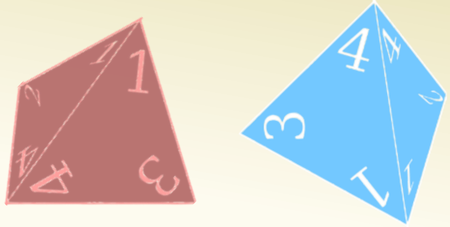
The joint PMF $p_{U,W}(u, w) = \Pr(U = u, W = w)$ is

$$p_{U,W}(u,w) = \begin{cases} 2/16, & (u,w) \in \Omega(U) \times \Omega(W) \ \text{ where } w > u \\ 1/16, & (u,w) \in \Omega(U) \times \Omega(W) \ \text{ where } w = u \\ 0, & \text{otherwise} \end{cases}$$

succinct form of

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

Example: Weirder Dice



Suppose I roll two fair 4-sided die independently. Let X be the value of the first die, and Y be the value of the second die. Let $U = \min(X, Y)$ and $W = \max(X, Y)$

Suppose we didn't know how to compute $\Pr(U = u)$ directly. Can we figure it out if we know $p_{U,W}(u, w)$?

$$p_U(u) = \begin{cases} 7/16, & u = 1 \\ 5/16, & u = 2 \\ 3/16, & u = 3 \\ 1/16, & u = 4 \end{cases}$$

Marginal PMF of U

$\Pr(U=1) = 7/16$

$\Pr(U=2) = 5/16$

$\Pr(U=3) = 3/16$

$\Pr(U=4) = 1/16$

U\W	1	2	3	4
1	1/16	2/16	2/16	2/16
2	0	1/16	2/16	2/16
3	0	0	1/16	2/16
4	0	0	0	1/16

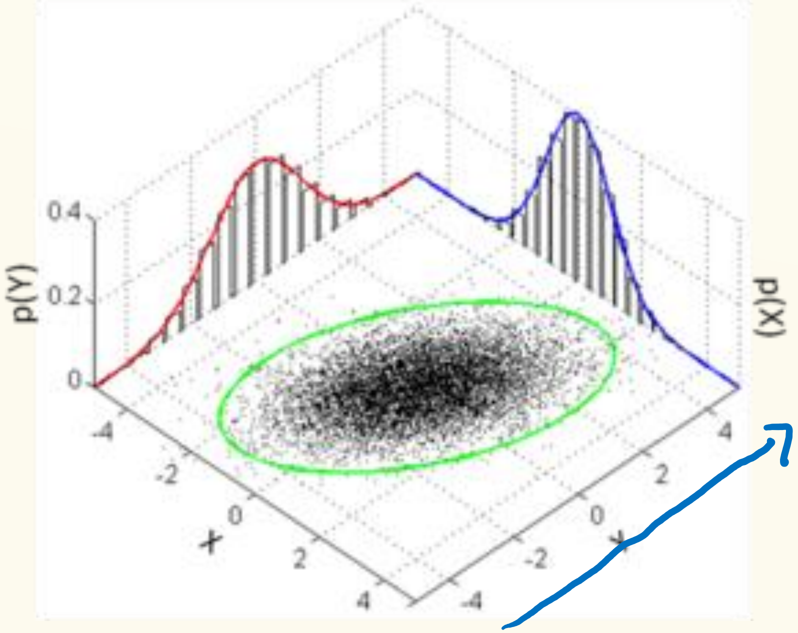
Marginal PMF

Definition. Let X and Y be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **marginal PMF** of X

$$\underline{p_X(a)} = \sum_{\underline{b\in\Omega(Y)}} \underline{p_{X,Y}(a,b)}$$

Similarly, $p_Y(b) = \sum_{a\in\Omega(X)} p_{X,Y}(a,b)$

Visual (for continuous X and Y)



Joint Expectation

$E[UWU]$

Definition. Let X and Y be discrete random variables and $p_{X,Y}(a,b)$ their joint PMF. The **expectation** of some function $g(x,y)$ with inputs X and Y

$$E[g(X,Y)] = \sum_{a\in\Omega(X)} \sum_{b\in\Omega(Y)} g(a,b)p_{X,Y}(a,b)$$

$$E[U^2W] = \sum_{a\in\Omega(U)} \sum_{b\in\Omega(W)} a^2 b \, p_{UW}(a,b)$$

Brain Break

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- Joint Distributions
  - Cartesian Products
  - Joint PMFs and Joint Range
  - Marginal Distribution
- Conditional Expectation and Law of Total Expectation

Conditional Expectation

Definition. Let X be a discrete random variable then the **conditional expectation** of X given event A is

$$E[X \mid A] = \sum_{x \in \Omega(X)} x \Pr(X = x \mid A)$$

Notes:

- Can be phrased as a "random variable version"

$$E[X \mid Y = y]$$

Recall: $Y = y$ is an event

- Linearity of expectation still applies here

$$E[aX + bY + c \mid A] = aE[X \mid A] + bE[Y \mid A] + c$$

Law of Total Expectation

LTP: $\Pr(X=x) = \sum_{i=1}^{n} \Pr(X=x \mid A_i)\Pr(A_i)$

Law of Total Expectation (event version). Let X be a random variable and let events $A_1, \dots, A_n$ partition the sample space. Then,

$$E[X] = \sum_{i=1}^{n} E[X|A_i]\Pr(A_i)$$

Law of Total Expectation (random variable version). Let X be a random variable and Y be a discrete random variable. Then,

$$E[X] = \sum_{y \in \Omega(Y)} E[X|Y = y]\Pr(Y = y)$$

Proof of Law of Total Expectation

Follows from Law of Total Probability and manipulating sums

$$E[X] = \sum_{x \in \Omega(X)} x \Pr(X = x)$$

$$= \sum_{x \in \Omega(X)} x \sum_{i=1}^{n} \Pr(X = x \mid A_i) \Pr(A_i)$$ (by LTP)

$$= \sum_{i=1}^{n} \Pr(A_i) \sum_{x \in \Omega(X)} x \Pr(X = x \mid A_i)]$$ (change order of sums)

$$= \sum_{i=1}^{n} \Pr(A_i) E[X \mid A_i]$$ (def of cond. expect.)

Example: Flipping Coins

If $Y=4$, easier to compute: $E[X|Y=4] = 4/2 = 2$

If $Y=y$: $E[X|Y=y] = y/2$

Suppose wanted to analyze flipping a random number of coins. Suppose someone gave us $Y \sim Poi(5)$ fair coins and we wanted to compute the expected number of heads X from flipping those coins.

$X \sim Bin(Y, 1/2)$

↑ Y is RV!

$$E[X] \overset{\text{LTE}}{=} \sum_{y=0}^{\infty} E[X|Y=y] \Pr(Y=y)$$

$$= \sum_{y=0}^{\infty} \frac{y}{2} \, e^{-5} \frac{5^y}{y!}$$

$$= 2.5$$

↑ by Wolfram Alpha

If $E[X]$ is hard to compute, can you condition on other events and use LTE?

Example: Computer Failures

Suppose your computer operates in a sequence of steps, and that at each step i your computer will fail with probability p (independently of other steps). Let X be the number of steps it takes your computer to fail. What is $E[X]$?

Not covered today, will discuss Monday.

Reference Sheet (with continuous RVs)

	Discrete	Continuous
Joint PMF/PDF	$p_{X,Y}(x,y) = P(X = x, Y = y)$	$f_{X,Y}(x,y) \neq P(X = x, Y = y)$
Joint CDF	$F_{X,Y}(x,y) = \sum_{t \le x} \sum_{s \le y} p_{X,Y}(t,s)$	$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(t,s) ds dt$
Normalization	$\sum_{x} \sum_{y} p_{X,Y}(x,y) = 1$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx dy = 1$
Marginal PMF/PDF	$p_X(x) = \sum_{y} p_{X,Y}(x,y)$	$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy$
Expectation	$E[g(X,Y)] = \sum_{x} \sum_{y} g(x,y) p_{X,Y}(x,y)$	$E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dx dy$
Conditional PMF/PDF	$p_{X\mid Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$	$f_{X\mid Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$
Conditional Expectation	$E[X \mid Y = y] = \sum_{x} x p_{X\mid Y}(x \mid y)$	$E[X \mid Y = y] = \int_{-\infty}^{\infty} x f_{X\mid Y}(x \mid y) dx$
Independence	$\forall x, y, p_{X,Y}(x,y) = p_X(x) p_Y(y)$	$\forall x, y, f_{X,Y}(x,y) = f_X(x) f_Y(y)$