CSE 312

Foundations of Computing II

Lecture 20: Continuity Correction & Distinct Elements

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

The CLT – Recap

Theorem. (Central Limit Theorem) $X_1, \ldots, X_n$ iid with mean μ and variance σ^2. Let $Y_n = \frac{X_1+\cdots+X_n - n\mu}{\sigma\sqrt{n}}$. Then,

$$\lim_{n\to\infty} Y_n \to \mathcal{N}(0,1)$$

One main application:

Use Normal Distribution to Approximate Y_n

No need to understand Y_n !!

Example – Y_n is binomial

We understand binomial, so we can see how well approximation works

We flip n independent coins, heads with probability $p = 0.75$.

X = # heads $\mu = \mathbb{E}(X) = 0.75n$ $\sigma^2 = \mathrm{Var}(X) = p(1-p)n = 0.1875n$

$\sim \mathcal{N}(\mu, \sigma^2)$

$\mathbb{P}(X \le 0.7n)$

n	exact	$\mathcal{N}(\mu, \sigma^2)$ approx
10	0.4744072	0.357500327
20	0.38282735	0.302788308
50	0.25191886	0.207108089
100	0.14954105	0.124106539
200	0.06247223	0.051235217
1000	0.00019359	0.000130365

Example – Naive Approximation

n

Fair coin flipped (independently) **40** times. Probability of **20** or **21** heads?

Exact. $\mathbb{P}(X \in \{20,21\}) = \left[\binom{40}{20} + \binom{40}{21}\right]\boxed{\left(\frac{1}{2}\right)^{40}} \approx \boxed{0.2448}$

Approx. X = # heads $\quad \mu = \mathbb{E}(X) = 0.5n = 20 \quad \sigma^2 = \mathrm{Var}(X) = 0.25n = 10$

$\sim \mathcal{N}(20, 10)$

$$\mathbb{P}(20 \le X \le 21) = \Phi\left(\frac{20-20}{\sqrt{10}} \le \frac{X-20}{\sqrt{10}} \le \frac{21-20}{\sqrt{10}}\right)$$

$$\approx \Phi\left(0 \le \frac{X-20}{\sqrt{10}} \le 0.32\right)$$

$$= \Phi(0.32) - \Phi(0) \approx \boxed{0.1241}$$

Example – Even Worse Approximation

n

Fair coin flipped (independently) **40** times. Probability of **20** heads?

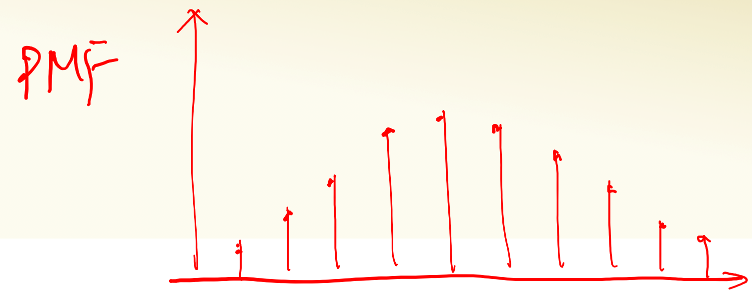
Exact. $\mathbb{P}(X = 20) = \binom{40}{20}\left(\frac{1}{2}\right)^{40} \approx \boxed{0.1254}$

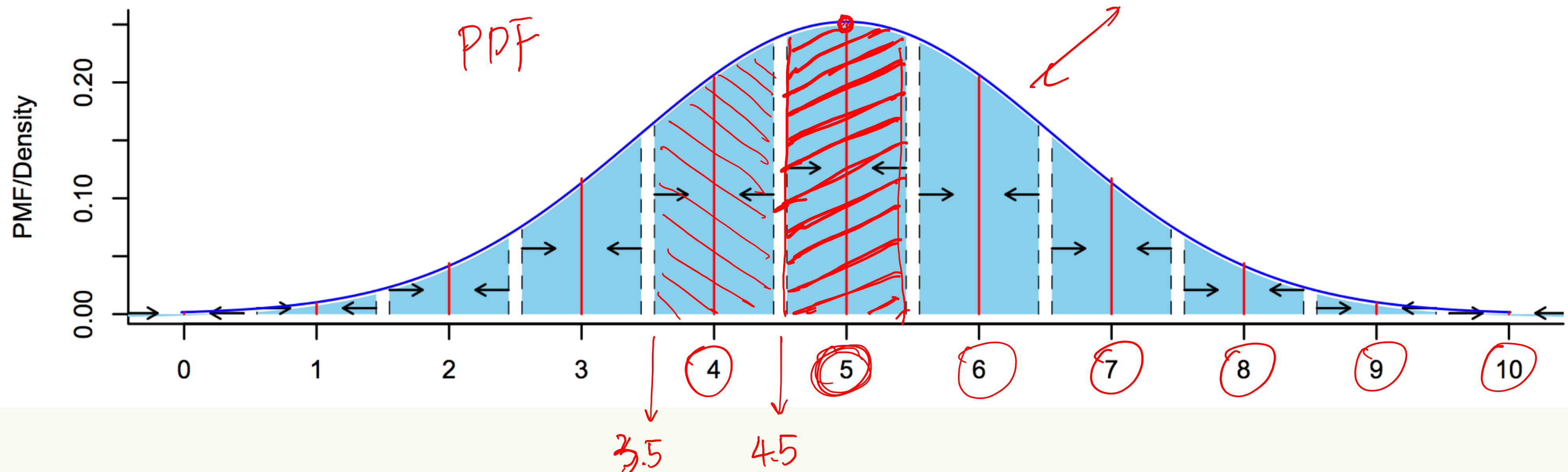
Approx. $\mathbb{P}(20 \le X \le 20) = 0$ 😢

$\sim \mathcal{N}(20, 10)$

Solution – Continuity Correction

Round to next integer!





To estimate probability that discrete RV lands in (integer) interval $\{a, \ldots, b\}$, compute probability continuous approximation lands in interval $[a - \frac{1}{2}, b + \frac{1}{2}]$

Example – Continuity Correction

Fair coin flipped (independently) **40** times. Probability of **20** or **21** heads?

Exact. $\mathbb{P}(X \in \{20,21\}) = \left[\binom{40}{20} + \binom{40}{21}\right]\left(\frac{1}{2}\right)^{40} \approx \boxed{0.2448}$

Approx. X = # heads $\quad \mu = \mathbb{E}(X) = 0.5n = 20 \quad \sigma^2 = \mathrm{Var}(X) = 0.25n = 10$

$$\mathbb{P}(19.5 \le X \le \cancel{21.5}) = \Phi\left(\frac{19.5-20}{\sqrt{10}} \le \frac{X-20}{\sqrt{10}} \le \frac{21.5-20}{\sqrt{10}}\right)$$

21.5

$$\approx \Phi\left(-0.16 \le \frac{X-20}{\sqrt{10}} \le 0.47\right)$$

$$= \Phi(-0.16) - \Phi(0.47) \approx \boxed{0.2452}$$

Example – Continuity Correction

Fair coin flipped (independently) **40** times. Probability of **20** heads?

Exact. $\mathbb{P}(X = 20) = \binom{40}{20}\left(\frac{1}{2}\right)^{40} \approx \boxed{0.1254}$

Approx. $\mathbb{P}(19.5 \le X \le \cancel{21.5}\overset{20.5}{}) = \Phi\left(\frac{19.5-20}{\sqrt{10}} \le \frac{X-20}{\sqrt{10}} \le \frac{20.5-20}{\sqrt{10}}\right)$

$$\approx \Phi\left(-0.16 \le \frac{X-20}{\sqrt{10}} \le 0.16\right)$$

$$= \Phi(-0.16) - \Phi(0.16) \approx \boxed{0.1272}$$

Application: Distinct Elements (code this in Pset 6)

Data mining – Stream Model

- In many data mining situations, the data is not known ahead of time.
  Examples: Google queries, Twitter or Facebook status updates
  Youtube video views
- In some ways, best to think of the data as an infinite stream that is non-stationary (distribution changes over time)
- Input elements (e.g. Google queries) enter/arrive one at a time.
  We cannot possibly store the stream.

Question: How do we make critical calculations about the data stream using a limited amount of memory?

Problem Setup

- Input: sequence of N elements $x_1, x_2, \ldots, x_N$ from a known universe U (e.g., 8-byte integers).
- Goal: perform a computation on the input, in a single left to right pass where
  - Elements processed in real time
  - Can't store the full data. => use minimal amount of storage while maintaining working "summary"

What can we compute?

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

- Some functions are easy:
  - Min
  - Max
  - Sum
  - Average

Today: Counting distinct elements

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

Application:

You are the content manager at YouTube, and you are trying to figure out the **distinct** view count for a video. How do we do that?

Note: A person can view their favorite videos several times, but they only count as 1 **distinct** view!

Other applications

- IP packet streams: How many distinct IP addresses or IP flows (source+destination IP, port, protocol)
  - * Anomaly detection, traffic monitoring
- Search: How many distinct search queries on Google on a certain topic yesterday
- Web services: how many distinct users (cookies) searched/browsed a certain term/item
  - * Advertising, marketing trends, etc.

Counting distinct elements

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

N = # of IDs in the stream = 11, m = # of distinct IDs in the stream = 5

Want to compute number of **distinct** IDs in the stream.

- *Naïve solution: As the data stream comes in, store all distinct IDs in a hash table.*
- *Space requirement O(m) , where m is the number of distinct IDs*

- *Consider the number of users of youtube, and the number videos on youtube. This is not feasible.*

Counting distinct elements

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

Want to compute number of **distinct** IDs in the stream.

- *How to do this without storing all the elements?*

Yet another super cool application of probability

Counting distinct elements

32, 12, 14, 32, 7, 12, 32, 7, 32, 12, 4

$y_1, y_2, y_3, y_1, y_4, y_2, y_1, y_4, y_1, y_2, y_5$

heuristic

Hash function $h: U \rightarrow [0,1]$

Assumption: For distinct values in U, the function maps to iid (independent and identically distributed) Unif(0,1) random numbers.

Important: if you were to feed in two equivalent elements, the function returns the **same** number.

- So m distinct elements → m iid uniform y_i's

Min of IID Uniforms

If $Y_1, \cdots, Y_m$ are iid Unif(0,1), where do we expect the points to end up?

In general, $E[\min(Y_1, \cdots, Y_m)] = \frac{1}{m+1}$

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session
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$$E[\min(Y_1)] = \frac{1}{1+1} = \frac{1}{2}$$

$m = 1$

0 1

$$E[\min(Y_1, Y_2)] = \frac{1}{2+1} = \frac{1}{3}$$

$m = 2$

0 1

$$E[\min(Y_1, \cdots, Y_4)] = \frac{1}{4+1} = \frac{1}{5}$$

$m = 4$

0 1

A super duper clever idea

If $Y_1, \cdots, Y_n$ are iid Unif(0,1), where do we expect the points to end up?

In general, $E[\min(Y_1, \cdots, Y_m)] = \frac{1}{m+1}$

Idea: $m = \frac{1}{E[\underbrace{\min(Y_1,\cdots,Y_m)}_{val}]} - 1$

Let's keep track of the value val of min of hash values,

and estimate m as Round $\left(\frac{1}{val} - 1\right)$

The Distinct Elements Algorithm

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow$ min {val, hash(x)}
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$

for $i = 1, \ldots, N$: **do** ▷ Loop through all stream elements
  update(x_i) ▷ Update our single float variable
return estimate() ▷ An estimate for n, the number of distinct elements.

Distinct Elements Example

Stream: 13, 25, 19, 25, 19, 19

Hashes:

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow$ min {val, hash(x)}
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$
for $i = 1, \ldots, N$: **do** ▹ Loop through all stream elements
  update(x_i) ▹ Update our single float variable
return estimate() ▹ An estimate for n, the number of distinct elements.

val = infty

Distinct Elements Example

Stream: 13, 25, 19, 25, 19, 19

Hashes: 0.51,

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow$ min {val, hash(x)}
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$
for $i = 1, \ldots, N$: **do** ▷ Loop through all stream elements
  update(x_i) ▷ Update our single float variable
return estimate() ▷ An estimate for n, the number of distinct elements.

val = infty

Distinct Elements Example

Stream: 13, 25, 19, 25, 19, 19

Hashes: 0.51,

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow$ min {val, hash(x)}
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$
for $i = 1, \ldots, N$: **do** ▹ Loop through all stream elements
  update(x_i) ▹ Update our single float variable
return estimate() ▹ An estimate for n, the number of distinct elements.

val = 0.51

Distinct Elements Example

Stream: 13, 25, 19, 25, 19, 19

Hashes: 0.51, 0.26,

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow$ min {val, hash(x)}
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$
for $i = 1, \ldots, N$: **do** ▷ Loop through all stream elements
  update(x_i) ▷ Update our single float variable
return estimate() ▷ An estimate for n, the number of distinct elements.

val = 0.26

Distinct Elements Example

Stream: 13, 25, 19, 25, 19, 19

Hashes: 0.51, 0.26, 0.79,

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow$ min {val, hash(x)}
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$
for $i = 1, \ldots, N$: **do** ▷ Loop through all stream elements
  update(x_i) ▷ Update our single float variable
return estimate() ▷ An estimate for n, the number of distinct elements.

val = 0.26

Distinct Elements Example

Stream: 13, 25, 19, 25, 19, 19

Hashes: 0.51, 0.26, 0.79, 0.26,

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow$ min {val, hash(x)}
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$
for $i = 1, \ldots, N$: **do** ▷ Loop through all stream elements
  update(x_i) ▷ Update our single float variable
return estimate() ▷ An estimate for n, the number of distinct elements.

val = 0.26

Distinct Elements Example

Stream: 13, 25, 19, 25, 19, 19

Hashes: 0.51, 0.26, 0.79, 0.26, 0.79,

Algorithm 2 Distinct Elements Operations

> **function** INITIALIZE()
>   val $\leftarrow \infty$
> **function** UPDATE(x)
>   val $\leftarrow$ min {val, hash(x)}
> **function** ESTIMATE()
>   **return** round $\left(\frac{1}{\text{val}} - 1\right)$
> **for** $i = 1, \ldots, N$: **do** ▷ Loop through all stream elements
>   update(x_i) ▷ Update our single float variable
> **return** estimate() ▷ An estimate for n, the number of distinct elements.

val = 0.26

Distinct Elements Example

Stream: 13, 25, 19, 25, 19, 19

Hashes: 0.51, 0.26, 0.79, 0.26, 0.79, 0.79

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow \min\{\text{val}, \text{hash}(x)\}$
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$

for $i = 1, \ldots, N$: **do** ▹ Loop through all stream elements
  update(x_i) ▹ Update our single float variable
return estimate() ▹ An estimate for n, the number of distinct elements.

val = 0.26

Distinct Elements Example

Stream: 13, 25, 19, 25, 19, 19

Hashes: 0.51, 0.26, 0.79, 0.26, 0.79, 0.79

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow$ min {val, hash(x)}
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$
for $i = 1, \ldots, N$: **do** ▷ Loop through all stream elements
  update(x_i) ▷ Update our single float variable
return estimate() ▷ An estimate for n, the number of distinct elements.

val = 0.26

Return

round(1/0.26 - 1) =
round(2.846) = 3

Diy: Distinct Elements Example II

Stream: 11, 34, 89, 11, 89, 23

Hashes: 0.5, 0.21, 0.94, 0.5, 0.94, 0.1

Algorithm 2 Distinct Elements Operations

function INITIALIZE()
  val $\leftarrow \infty$
function UPDATE(x)
  val $\leftarrow$ min {val, hash(x)}
function ESTIMATE()
  return round $\left(\frac{1}{\text{val}} - 1\right)$
for $i = 1, \ldots, N$: **do** ▹ Loop through all stream elements
  update(x_i) ▹ Update our single float variable
return estimate() ▹ An estimate for n, the number of distinct elements.

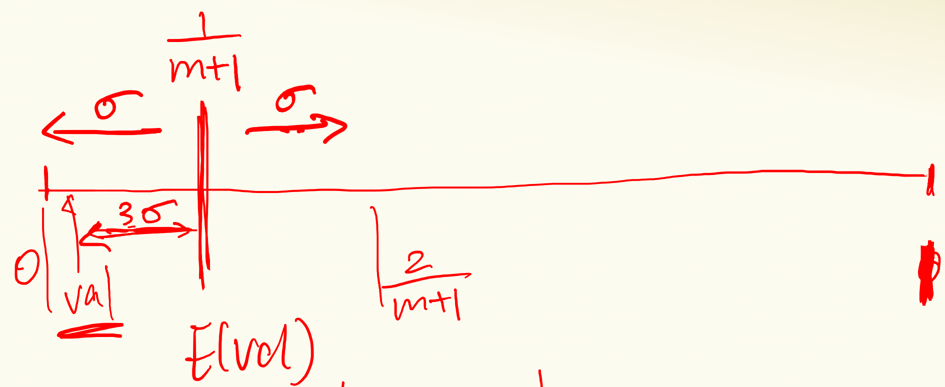
val = 0.1

Return= 9

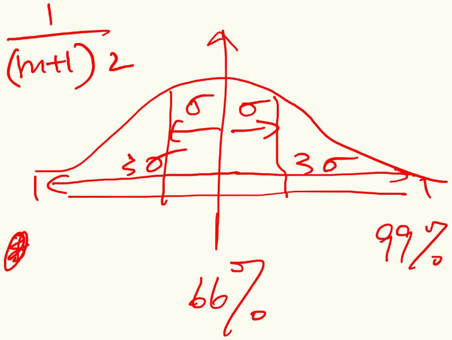
Problem

$\text{val} = \min(Y_1, \cdots, Y_m)$

$E[val] = \dfrac{1}{m+1}$ Fact



$\frac{1}{m+1} - 3 \cdot \frac{1}{(m+1)6} = \frac{1}{(m+1)2}$



Algorithm:

Track $val = \min\big(h(X_1), \cdots, h(X_N)\big) = \min(Y_1, \cdots, Y_m)$

estimate m = 1/val -1

Round$\left(\frac{1}{val} - 1\right)$

But, val is not E[val]! How far is val from E[val]?

$\text{Var}[val] \approx \dfrac{1}{(m+1)^2}$ $\boxed{\sigma = \frac{1}{m+1}}$

How can we reduce the variance?

Idea: Repetition to reduce variance!

Use k **independent** hash functions $h^1, h^2, \cdots h^k$

Keep track of k independent min hash values

$$val^1 = \min\left(h^1(x_1), \cdots, h^1(x_N)\right) = \min(Y_1^1, \cdots, Y_m^1)$$

$$val^2 = \min\left(h^2(x_1), \cdots, h^2(x_N)\right) = \min(Y_1^2, \cdots, Y_m^2)$$

… …

$$val^k = \min\left(h^k(x_1), \cdots, h^k(x_N)\right) = \min(Y_1^k, \cdots, Y_m^k)$$

$val = \frac{1}{k}\Sigma_i val_i$, Estimate $m = \frac{1}{val} - 1$

e.g. $k = 36$.

$\sigma = \frac{1}{(m+1)6}$.

$E(val) = \frac{1}{m+1}$

$Var(val) = \frac{\sum_i Var(val^i)}{k^2}$

$= \frac{var(val^1)}{k}$