Announcement

• HW1 out 11:59pm Friday and is due 11:59pm next Friday.
• In general, every HW is due one week after it’s released.
• Later HWs will be released on different days of the week. We will put up the schedule in the schedule page of the course website.
Last Class: Counting
• Sequential process
• Product rule
• Representation of the problem is important (creative part)

Today: More Counting
• Permutations and Combinations
Note: Sequential process works even if the set of options are different at each point

“How many sequences in \(1,2,3\)^3 with no repeating elements?”

\[
\begin{array}{c}
3^3 \\
3 \times 2 \times 1 = 6
\end{array}
\]
Factorial

“How many ways to order elements in \( S \), where \(|S| = n\)?”

Answer: \( n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1 \)

**Definition.** The factorial function is

\[ n! = n \times (n - 1) \times \cdots \times 2 \times 1 \]

**Note:** \( 0! = 1 \)

**Theorem.** (Stirling’s approximation)

\[ \sqrt{2\pi \cdot n^{n+\frac{1}{2}} \cdot e^{-n}} \leq n! \leq e \cdot n^{n+\frac{1}{2}} \cdot e^{-n} \]

\[ = 2.5066 \quad = 2.7183 \]

**Huge:** Grows exponentially in \( n \)
Distinct Letters

“How many sequences of 5 distinct alphabet letters from \{A, B, \ldots, Z\}?”

E.g., AZURE, BINGO, TANGO. But not: STEVE, SARAH

Answer: \(26 \times 25 \times 24 \times 23 \times 22 = 7893600\)
In general

Aka: $k$-permutations

Fact. # of $k$-element sequences of distinct symbols from $n$-element set is

$$P(n, k) = n \times (n - 1) \times \cdots \times (n - k + 1) = \frac{n!}{(n - k)!}$$
Number of Subsets

“How many size-5 subsets of \{A, B, ..., Z\}?”


Difference from \(k\)-permutations: NO ORDER

Different sequences: TANGO, OGNAT, ATNGO, NATGO, ONATG ...

Number of Subsets – Idea

Consider a sequential process:
1. Choose a subset $S \subseteq \{A, B, \ldots, Z\}$ of size $|S| = 5$
   e.g. $S = \{A, G, N, O, T\}$
2. Choose a permutation of letters in $S$
   e.g., TANGO, AGNOT, NAGOT, GOTAN, GOATN, NGOAT, ...

Outcome: A sequences of 5 distinct letters from $\{A, B, \ldots, Z\}$

$$\frac{26!}{21!5!} = 65780$$
Number of Subsets – Binomial Coefficient

**Fact.** The number of subsets of size \( k \) of a set of size \( n \) is

\[
C(n, k) = \binom{n}{k} = \frac{n!}{k! (n-k)!}
\]

Binomial coefficient (verbalized as “\( n \) choose \( k \)”)

**Notation:** \( \binom{S}{k} = \text{all } k \text{-element subsets of } S \)

[also called **combinations**]

\[
= \left\{ X : X \subseteq S, \; |X| = k \right\}
\]

\[
\left| \binom{S}{k} \right| = \binom{|S|}{k}
\]
Example – Counting Paths

“How many shortest paths from Gates to Starbucks?”
Example – Counting Paths

How do we represent a path?
Example – Counting Paths

\[ \text{Path} \in \{↑, →\}^8 \]

\[ \#↑'s = \#→'s = 4 \]

Poll:
A. \( 2^8 \)
B. \( \frac{8!}{4!} \)
C. \( \binom{8}{4} = \frac{8!}{4!4!} \)
D. No idea

\[ \binom{9}{4} = \frac{9!}{4!5!} = \frac{9}{5} \]
Example – Sum of integers

“How many solutions \((x_1, \ldots, x_k)\) such that \(x_1, \ldots, x_k \geq 0\) and \(\sum_{i=1}^{k} x_i = n?\)”

Example: \(k = 3, n = 5\)

\((3, 2, 0)\)

\((0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots\)

Hint: we can represent each solution as a binary string.
Example – Sum of integers

Example: $k = 3, n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), ...$

Clever representation of solutions

$(3,1,1)$

$(2,1,2)$

$(1,0,4)$

$1110101$

$1101011$

$1001111$
Example – Sum of integers

Example: \( k = 3, \ n = 5 \)

\[
\# \text{sols} = \# \text{strings from } \{0,1\}^7 \text{ w/ exactly two } 0\text{s} = \binom{7}{2} = 21
\]

Clever representation of solutions

\[
\begin{align*}
(3,1,1) & : 1110101 \\
(2,1,2) & : 1101011 \\
(1,0,4) & : 1001111
\end{align*}
\]
Example – Sum of integers

“How many solutions \((x_1, \ldots, x_k)\) such that \(x_1, \ldots, x_k \geq 0\) and \(\sum_{i=1}^{k} x_i = n\)?”

\[
\# \text{sols} = \# \text{strings from } \{0,1\}^{n+k-1} \text{ w/ } k - 1 \text{ os}
= \frac{(n + k - 1)}{k - 1}
\]

After a change in representation, the problem magically reduces to counting combinations.
Example – Word Permutations

“How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, ...

Guess: 7! Correct?!

No! e.g., swapping two T’s lead both to SEATTLE
swapping two E’s lead both to SEATTLE

Counted as separate permutations, but they lead to the same word.
Example – Word Permutations

“How many ways to re-arrange the letters in the word SEATTLE? STALEET, TEALEST, LASTTEE, …

\[
\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1
\]

Locations of T’s
Locations of E’s
Location of S
Location of L
Location of A

Locations of T’s
Locations of E’s
Location of S
Location of L
Location of A
Example II – Word Permutations

“How many ways to re-arrange the letters in the word SEATTLE?”

STALEET, TEALEST, LASTTEE, ...

\[ \binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1 = \frac{7!}{2! \times 5!} \times \frac{5!}{2! \times 3!} \times 3! \]

\[ = \frac{7!}{2! \times 2!} = 1260 \]

Another interpretation:

Arrange the 7 letters as if they were distinct. Then divide by 2! to account for 2 duplicate T’s, and divide by 2! again for 2 duplicate E’s.
Binomial Coefficient – Many interesting and useful properties

\[ \binom{n}{k} = \frac{n!}{k! (n-k)!} \]

\[ \binom{n}{n} = 1 \quad \binom{n}{1} = n \quad \binom{n}{0} = 1 \]

**Fact.** \( \binom{n}{k} = \binom{n}{n-k} \)

Symmetry in Binomial Coefficients

**Fact.** \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

Pascal’s Identity

**Fact.** \( \sum_{k=0}^{n} \binom{n}{k} = 2^n \)

Follows from Binomial theorem

(Next lecture)
Symmetry in Binomial Coefficients

Fact. \( \binom{n}{k} = \binom{n}{n-k} \)

This is called an Algebraic proof, i.e., Prove by checking algebra

Proof. \( \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} \binom{n}{n-k} \)

Why??
Symmetry in Binomial Coefficients – A different proof

Fact. \( \binom{n}{k} = \binom{n}{n-k} \)

Two equivalent ways to choose \( k \) out of \( n \) objects (unordered)
1. Choose which \( k \) elements are included
2. Choose which \( n - k \) elements are excluded

\[
\begin{align*}
\binom{4}{1} &= 4 = \binom{4}{3} \\
\end{align*}
\]
Symmetry in Binomial Coefficients – A different proof

Fact. \( \binom{n}{k} = \binom{n}{n-k} \)

Two equivalent ways to choose \( k \) out of \( n \) objects (unordered)
1. Choose which \( k \) elements are included
2. Choose which \( n-k \) elements are excluded

This is called a combinatorial argument/proof
- Let \( S \) be a set of objects
- Show how to count \( |S| \) one way \( \Rightarrow |S| = \binom{N}{m} \)
- Show how to count \( |S| \) another way \( \Rightarrow |S| = \binom{n-k}{m} \)
combinatorial argument/proof
- Elegant
- Simple
- Intuitive

Algebraic argument
- Brute force
- Less Intuitive
Pascal’s Identities

**Fact.** \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

How to prove Pascal’s identity?

**Algebraic argument:**

\[
\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)! (n-k)!} + \frac{(n-1)!}{k! (n-1-k)!} \\
= 20 \text{ years later...} \\
= \frac{n!}{k! (n-k)!} \\
= \binom{n}{k}
\]

Hard work and not intuitive

Let’s see a combinatorial argument
Disjoint Sets

Sometimes, we want $|S|$, and $S = A \cup B$

**Fact.** $|A \cup B| = |A| + |B|$
Example – Binomial Identity

Fact. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

\(|S| = |A| + |B| \quad S = A \cup B\)

**S:** the set of size \( k \) subsets of \([n] = \{1, 2, \ldots, n\}\) \( \Rightarrow \quad |S| = \binom{n}{k} \)

e.g.: \( n = 4, S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\} \)

**A:** the set of size \( k \) subsets of \([n]\) including \( n \)
\[ A = \{\{1,4\}, \{2,4\}, \{3,4\}\} \]

**B:** the set of size \( k \) subsets of \([n]\) NOT including \( n \)
\[ B = \{\{1,2\}, \{1,3\}, \{2,3\}\} \]
Example – Binomial Identity

Fact. \( \binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \)

|S| |A| |B| \( S = A \cup B \)

\( S \): the set of size \( k \) subsets of \([n] = \{1, 2, \ldots, n\}\)

\( A \): the set of size \( k \) subsets of \([n]\) including \( n \)

\( B \): the set of size \( k \) subsets of \([n]\) NOT including \( n \)

\( n \) is in set, need to choose \( k - 1 \) elements from \([n - 1]\)

\( |A| = \binom{n-1}{k-1} \)

\( n \) not in set, need to choose \( k \) elements from \([n - 1]\)

\( |B| = \binom{n-1}{k} \)
Quick Summary

- **K-sequences**: How many length k sequence over alphabet of size n?
  - Product rule $\rightarrow n^k$

- **K-permutations**: How many length k sequence over alphabet of size n, without repetition?
  - Permutation $\rightarrow \frac{n!}{(n-k)!}$

- **K-combinations**: How many size k subset of a set of size n (without repetition and without order)?
  - Combination $\rightarrow \binom{n}{k} = \frac{n!}{k!(n-k)!}$