CSE 312

Foundations of Computing II

Lecture 2: Permutation and Combinations

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Announcement

- HW1 out 11:59pm Friday and is due 11:59pm next Friday.
- In general, every HW is due one week after it's released.
- Later HWs will be released on different days of the week. We will put up the schedule in the schedule page of the course website.

Last Class: Counting

- Sequential process
- Product rule
- Representation of the problem is important (creative part)

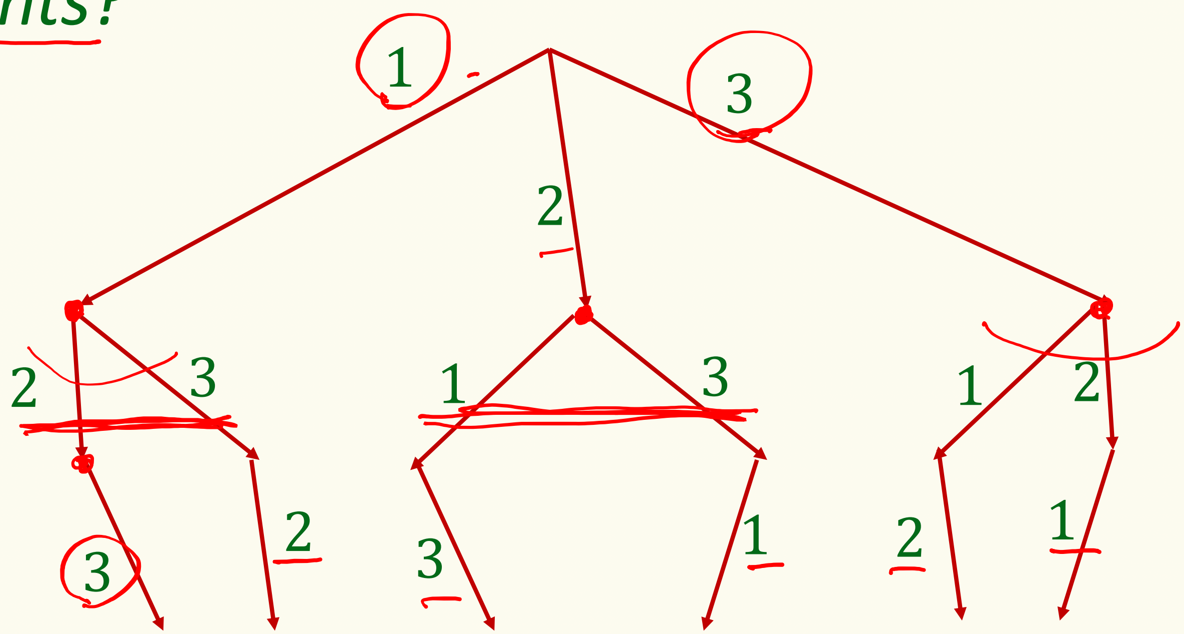
Today: More Counting

- Permutations and Combinations

Note: Sequential process works even if the set of options are different at each point

3^3

"How many sequences in $\{1,2,3\}^3$ *~~with no repeating elements~~?"*



$\boxed{3} \times \boxed{2} \times \boxed{1} = \boxed{6}$

Factorial

"How many ways to order elements in S, where $|S| = n$?" **Permutations**

Answer = $n \times \underbrace{(n-1)\times(n-2)\times\cdots\times 2\times 1}$

Definition. The **factorial function** is

$$n! = n\times(n-1)\times\cdots\times 2\times 1$$

Note: $0! = 1$

$n^n \cdot e^{-n} = \left(\frac{n}{e}\right)^{n}$ $n > e$

Theorem. (Stirling's approximation)

$$\underbrace{\sqrt{2\pi}}_{=2.5066}\cdot \boxed{n^{n}}^{+\frac{1}{2}}\cdot \boxed{e^{-n}} \le n! \le \underbrace{e}_{=2.7183}\cdot \boxed{n^{n}}^{+\frac{1}{2}}\cdot \boxed{e^{-n}}$$

Huge: Grows exponentially in n

Distinct Letters

"How many sequences of 5 distinct alphabet letters from $\{A, B, \dots, Z\}$*?"*

E.g., AZURE, BINGO, TANGO. But not: STEVE, SARAH

Answer: $26 \times 25 \times 24 \times 23 \times 22 = 7893600$

In general

Aka: k-permutations

Fact. # of k-element sequences of distinct symbols from n-element set is

$$P(n,k) = \underbrace{n \times (n-1) \times \cdots \times (n-k+1)}_{k} = \frac{n!}{(n-k)!}$$

Number of Subsets

*"How many size-5 **subsets** of* $\{A, B, \ldots, Z\}$*?"*

E.g., {A,Z,U,R,E}, {B,I,N,G,O}, {T,A,N,G,O}. But not: {S,T,E,V}, {S,A,R,H},…

Difference from k-permutations: NO ORDER

Different sequences: TANGO, OGNAT, ATNGO, NATGO, ONATG …

Same set: {T,A,N,G,O}, {O,G,N,A,T}, {A,T,N,G,O}, {N,A,T,G,O}, {O,N,A,T,G}… …

Number of Subsets – Idea

Consider a sequential process:

1. Choose a subset $S \subseteq \{A, B, \dots, Z\}$ of size $|S| = 5$
   e.g. $S = \{A, G, N, O, T\}$
2. Choose a permutation of letters in S
   e.g., *TANGO, AGNOT, NAGOT, GOTAN, GOATN, NGOAT, …*

Outcome: A sequences of 5 distinct letters from $\{A, B, \dots, Z\}$

$$\boxed{???} \times \boxed{5!} = \boxed{\frac{26!}{21!}}$$

$P(26,5) = \frac{26!}{21!}$

$$??? = \frac{26!}{21!\,5!} = 65780$$

Number of Subsets – Binomial Coefficient

Fact. The number of subsets of size k of a set of size n is

$$C(n,k) = \binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \binom{26}{5}\ \checkmark$$

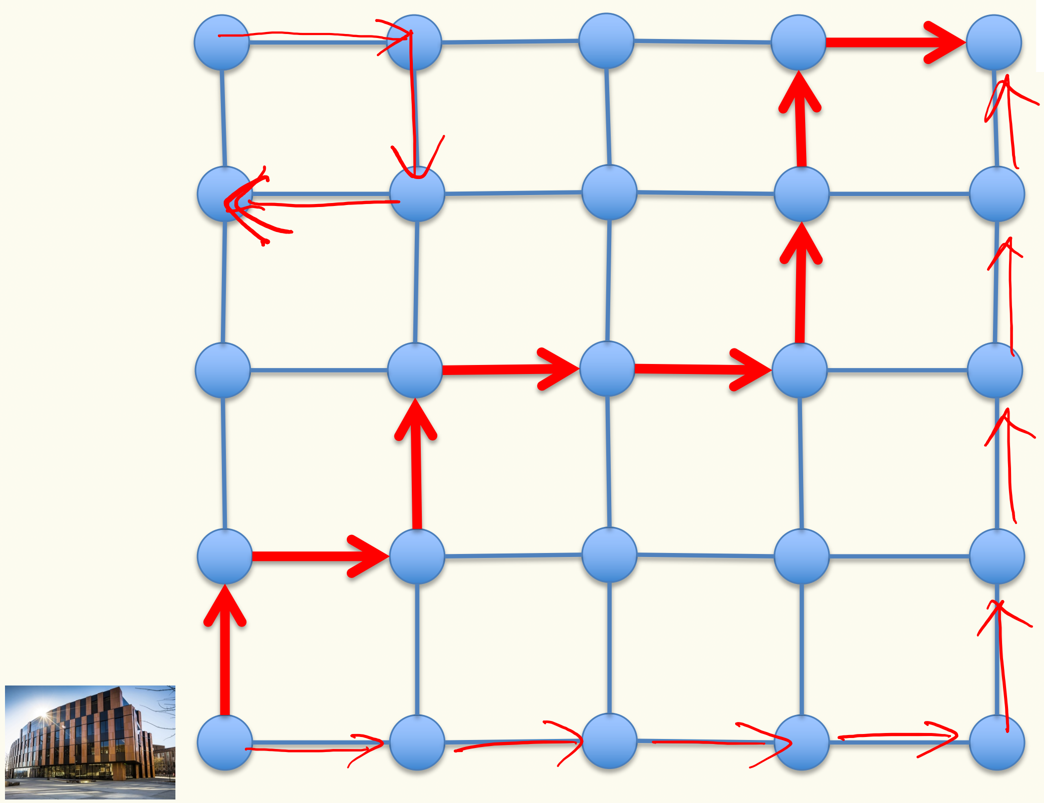
Binomial coefficient (verbalized as "n choose k")

Notation: $\binom{S}{k}$ = all k-element subsets of S $\qquad \left|\binom{S}{k}\right| = \binom{|S|}{k}$

[also called **combinations**]

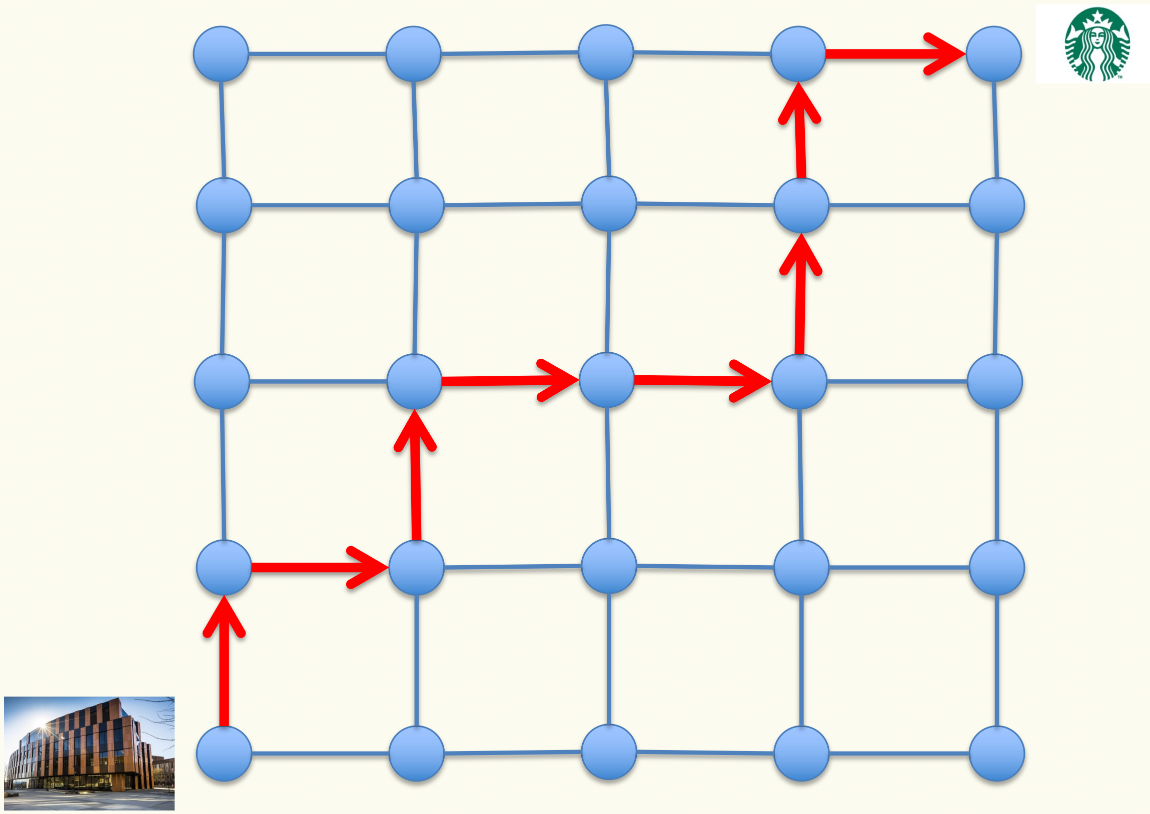
$= \{X : X \subseteq S,\ |X| = k\}$

Example – Counting Paths



“How many shortest paths from Gates to Starbucks?”

Example – Counting Paths



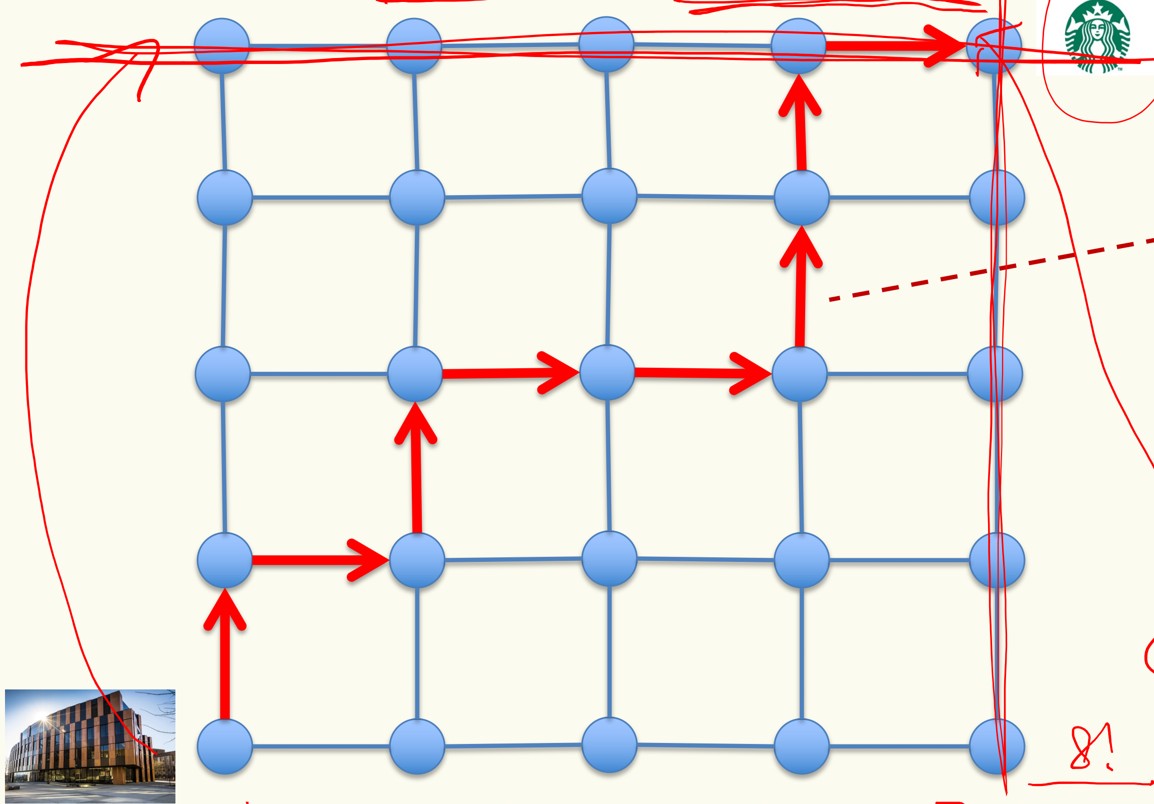
How do we represent a path?

Example – Counting Paths

$\binom{8}{4}4!$

↑ ↑ ↑ ↑

$\underset{1}{\rightarrow}\ \underset{2}{\rightarrow}\ \underset{3}{\rightarrow}\ \underset{4}{\rightarrow}$



$\frac{8!}{4!}$

$\{1, 3, 6, 7\}$

$\{1 \ldots 8\}$

$$\text{Path} \in \{\uparrow, \rightarrow\}^8$$

$(\uparrow, \rightarrow, \uparrow, \rightarrow, \rightarrow, \uparrow, \uparrow, \rightarrow)$

$\#\uparrow\text{'s} = \#\rightarrow\text{'s} = 4$

Poll:

A. 2^8

B. $\frac{8!}{4!}$

C. $\binom{8}{4} = \frac{8!}{4!4!}$

D. No idea

$\frac{8!}{4!}$ $\quad 8!$

$\binom{9}{4} = \binom{9}{5}$

Example – Sum of integers

"How many solutions $(x_1, \ldots, x_k)$ *such that* $x_1, \ldots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

Example: $k = 3$, $n = 5$ $(3,2,0)$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots$

Hint: we can represent each solution as a binary string.

Example – Sum of integers

1111001

(4, 0, 1)

Example: $k = 3$, $n = 5$

$(0,0,5), (5,0,0), (1,0,4), (2,1,2), (3,1,1), (2,3,0), \ldots$

$\# 1 = n$ $\quad$ $\# 1 = 5$

$\# 0 = k-1$ $\quad$ $\# 0 = 2 = 3-1$

Clever representation of solutions

(3,1,1)	(2,1,2)	(1,0,4)
↓	↓	↓
1 1 1 0 1 0 1	1 1 0 1 0 1 1	1 0 0 1 1 1 1

Example – Sum of integers

Example: $k = 3$, $n = 5$

\# sols = # strings from $\{0,1\}^7$ w/ exactly two 0s $= \binom{7}{2} = 21$

Clever representation of solutions

$(3,1,1)$ $\downarrow$ $1\ 1\ 1\ 0\ 1\ 0\ 1$

$(2,1,2)$ $\downarrow$ $1\ 1\ 0\ 1\ 0\ 1\ 1$

$(1,0,4)$ $\downarrow$ $1\ 0\ 0\ 1\ 1\ 1\ 1$

Example – Sum of integers

"How many solutions $(x_1, \dots, x_k)$ *such that* $x_1, \dots, x_k \geq 0$ *and* $\sum_{i=1}^{k} x_i = n$*?"*

\# sols = # strings from $\{0,1\}^{n+k-1}$ w/ $k-1$ 0s

$$= \binom{n+k-1}{k-1}$$

After a change in representation, the problem magically reduces to counting combinations.

Example – Word Permutations

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, ...

Guess: 7!   Correct?!

No! e.g., swapping two T's lead both to *SEATTLE*
swapping two E's lead both to *SEATTLE*

Counted as separate permutations, but they lead to the same word.

Example – Word Permutations

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

Location of S

Location of A

$$\binom{7}{2} \times \binom{5}{2} \times \underbrace{3 \times 2 \times 1}_{3!}$$

T E S T A L E

Location of L

Locations of E's

Locations of T's

Example II – Word Permutations

"How many ways to re-arrange the letters in the word SEATTLE?

STALEET, TEALEST, LASTTEE, …

$$\binom{7}{2} \times \binom{5}{2} \times 3 \times 2 \times 1 = \frac{7!}{2!\,\cancel{5!}} \times \frac{\cancel{5!}}{2!\,\cancel{3!}} \times \cancel{3!}$$

$$= \frac{7!}{2!\,2!} = 1260$$

Another interpretation:

Arrange the 7 letters as if they were distinct. Then divide by 2! to account for 2 duplicate T's, and divide by 2! again for 2 duplicate E's.

Binomial Coefficient – Many interesting and useful properties

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \binom{n}{n} = 1 \qquad \binom{n}{1} = n \qquad \binom{n}{0} = 1$$

$\emptyset$

Fact. $\binom{n}{k} = \binom{n}{n-k}$ **Symmetry in Binomial Coefficients**

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$ **Pascal's Identity**

Fact. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ **Follows from Binomial theorem (Next lecture)**

Symmetry in Binomial Coefficients

Fact. $\binom{n}{k} = \binom{n}{n-k}$

This is called an Algebraic proof,
i.e., Prove by checking algebra

Proof. $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n!}{(n-k)!k!} = \binom{n}{n-k}$

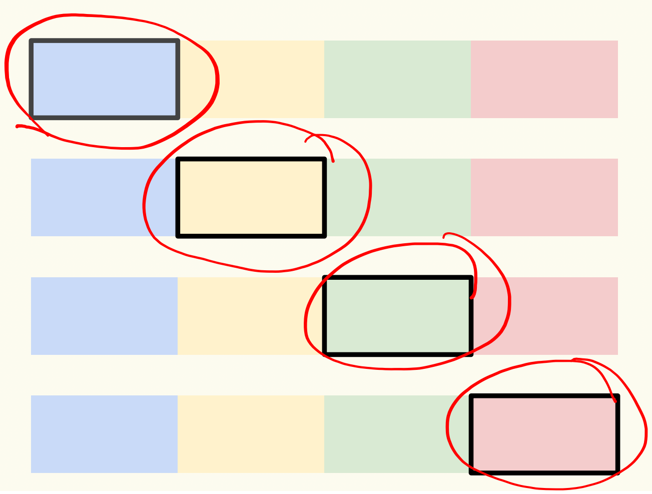
Why??

Symmetry in Binomial Coefficients – A different proof

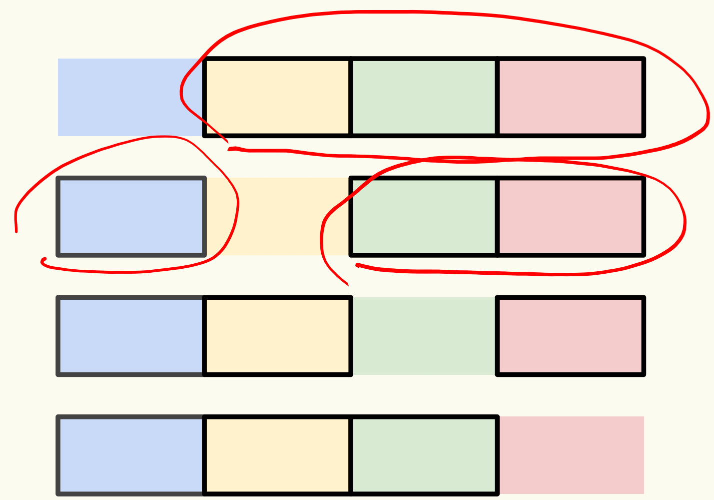
Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n-k$ elements are excluded



$$\binom{4}{1} = 4 = \binom{4}{3}$$



Symmetry in Binomial Coefficients – A different proof

$\forall n, k$ $0 \leq k \leq n$

Fact. $\binom{n}{k} = \binom{n}{n-k}$

Two equivalent ways to choose k out of n objects (unordered)

1. Choose which k elements are included
2. Choose which $n - k$ elements are excluded

This is called a combinatorial argument/proof

- Let S be a set of objects
- Show how to count $|S|$ one way => $|S| = N$
- Show how to count $|S|$ another way => $|S| = m$

$\binom{n}{k}$

$\shortparallel$

$\binom{n}{n-k}$

combinatorial argument/proof

- Elegant
- Simple
- Intuitive

Algebraic argument

- Brute force
- Less Intuitive

Pascal's Identities

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

How to prove Pascal's identity?

Algebraic argument:

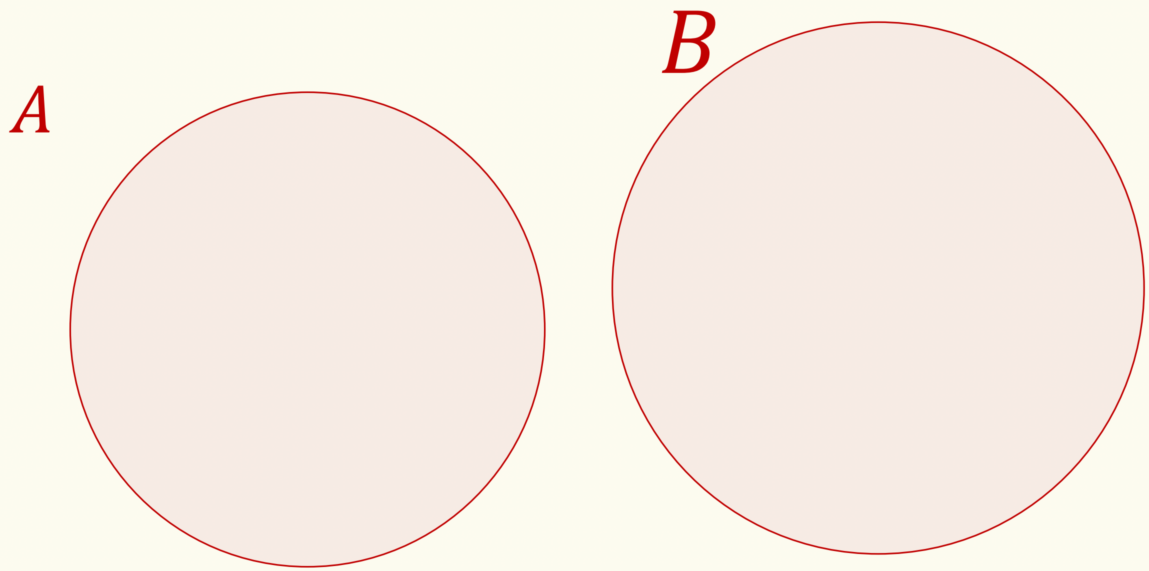
$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-1-k)!} \\ &= 20\ years\ later\ \ldots \\ &= \frac{n!}{k!\,(n-k)!} \\ &= \binom{n}{k}\end{aligned}$$

Hard work and not intuitive

Let's see a combinatorial argument

Disjoint Sets

Sometimes, we want $|S|$, and $S = A \cup B$



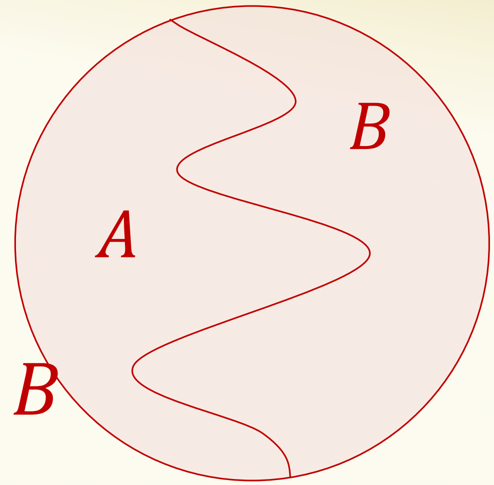
Fact. $|A \cup B| = |A| + |B|$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$|S| \quad = \quad |A| \quad + \quad |B|$$

$S = A \cup B$



S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$ ➔ $|S| = \binom{n}{k}$

e.g.: $n = 4, S = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$

A: the set of size k subsets of $[n]$ including n

$$A = \{\{1,4\}, \{2,4\}, \{3,4\}\}$$

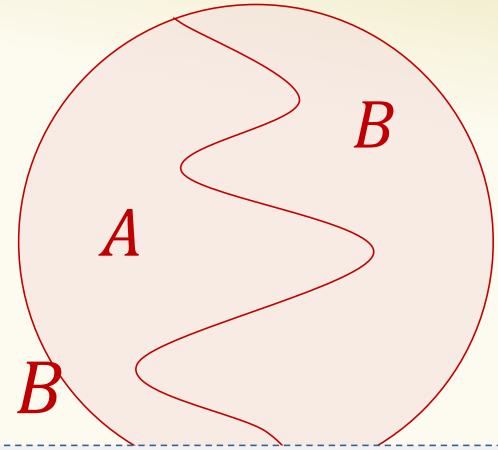
B: the set of size k subsets of $[n]$ NOT including n

$$B = \{\{1,2\}, \{1,3\}, \{2,3\}\}$$

Example – Binomial Identity

Fact. $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

$$\underset{|S|}{\binom{n}{k}} \qquad \underset{|A|}{\binom{n-1}{k-1}} \qquad \underset{|B|}{\binom{n-1}{k}}$$



$S = A \cup B$

S: the set of size k subsets of $[n] = \{1, 2, \cdots, n\}$

A: the set of size k subsets of $[n]$ including n

n is in set, need to choose $k-1$ elements from $[n-1]$

$$|A| = \binom{n-1}{k-1}$$

B: the set of size k subsets of $[n]$ NOT including n

n not in set, need to choose k elements from $[n-1]$

$$|B| = \binom{n-1}{k}$$

Quick Summary

- K-sequences: How many length k sequence over alphabet of size n?
  - Product rule ➔ n^K
- K-permutations: How many length k sequence over alphabet of size n, without repetition?
  - Permutation ➔ $\frac{n!}{(n-k)!}$
- K-combinations: How many size k subset of a set of size n (without repetition and without order)?
  - Combination ➔ $\binom{n}{k} = \frac{n!}{k!(n-k)!}$