CSE 312

Foundations of Computing II

Lecture 15: Exponential and Normal Distribution

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Review – Continuous RVs

Probability ~~Density~~ Mass Function (PDF).

$f: \mathbb{R} \to \mathbb{R}$ s.t.

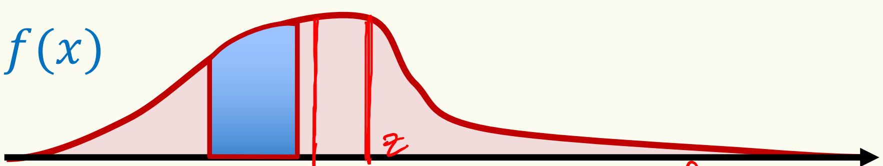
- $f(x) \geq 0$ for all $x \in \mathbb{R}$
- $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$

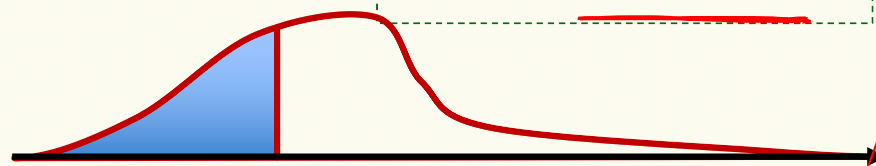
Cumulative Density Function (CDF).

$$F(y) = \int_{-\infty}^{y} f(x)\,\mathrm{d}x$$

Theorem. $f(x) = \frac{dF(x)}{dx}$

$f(x)$





$P(X=x)=0$ $\quad f_X(x) > 0 > 1$

Density ≠ Probability !

$$\boxed{\mathbb{P}(X \in [a,b]) = \int_a^b f_X(x)\mathrm{d}x}$$

$$\sum_{x\in\Omega(X) \wedge x\in[a,b]} P(X=x) = F_X(b) - F_X(a)$$

$$F(y) = \mathbb{P}(X \leq y)$$

Expectation of a Continuous RV

$\int f_1(x) + f_2(x)\, dx$

$= \int f_1(x)\, dx$

$+ \int f_2(x)\, dx$

Definition. The **expected value** of a continuous RV X is defined as

$$E(X) = \sum_{x \in \Omega(X)} P(X=x) \cdot x \qquad \mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Fact. $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

Definition. The **variance** of a continuous RV X is defined as

$$Var(X) = \sum_{x \in \Omega(X)} P(X=x) \cdot (x - E(X))^2$$

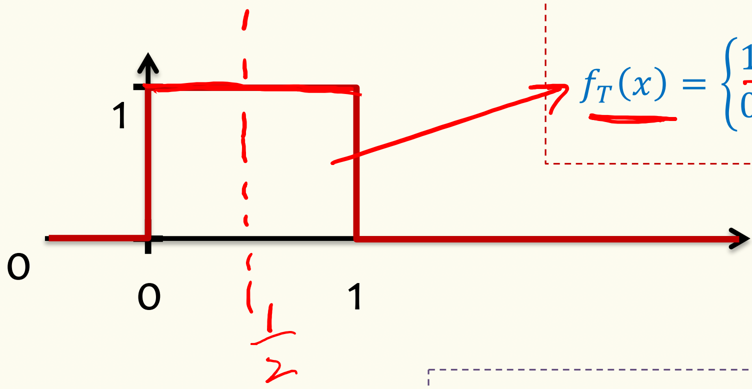
$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot \left(x - \mathbb{E}(X)\right)^2 \mathrm{d}x = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

Expectation of a Continuous RV

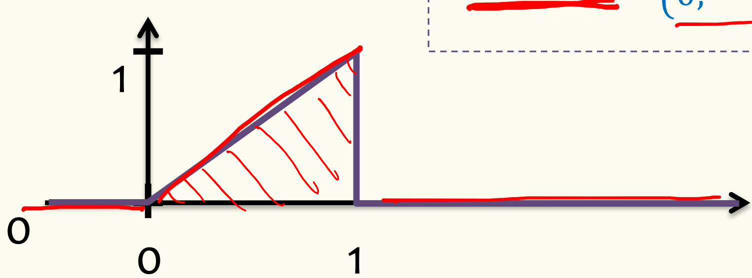
Definition.

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} \boxed{f_X(x) \cdot x}\, \mathrm{d}x$$

Example. $T \sim \mathrm{Unif}(0,1)$



$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$f_T(x) \cdot x = \begin{cases} x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

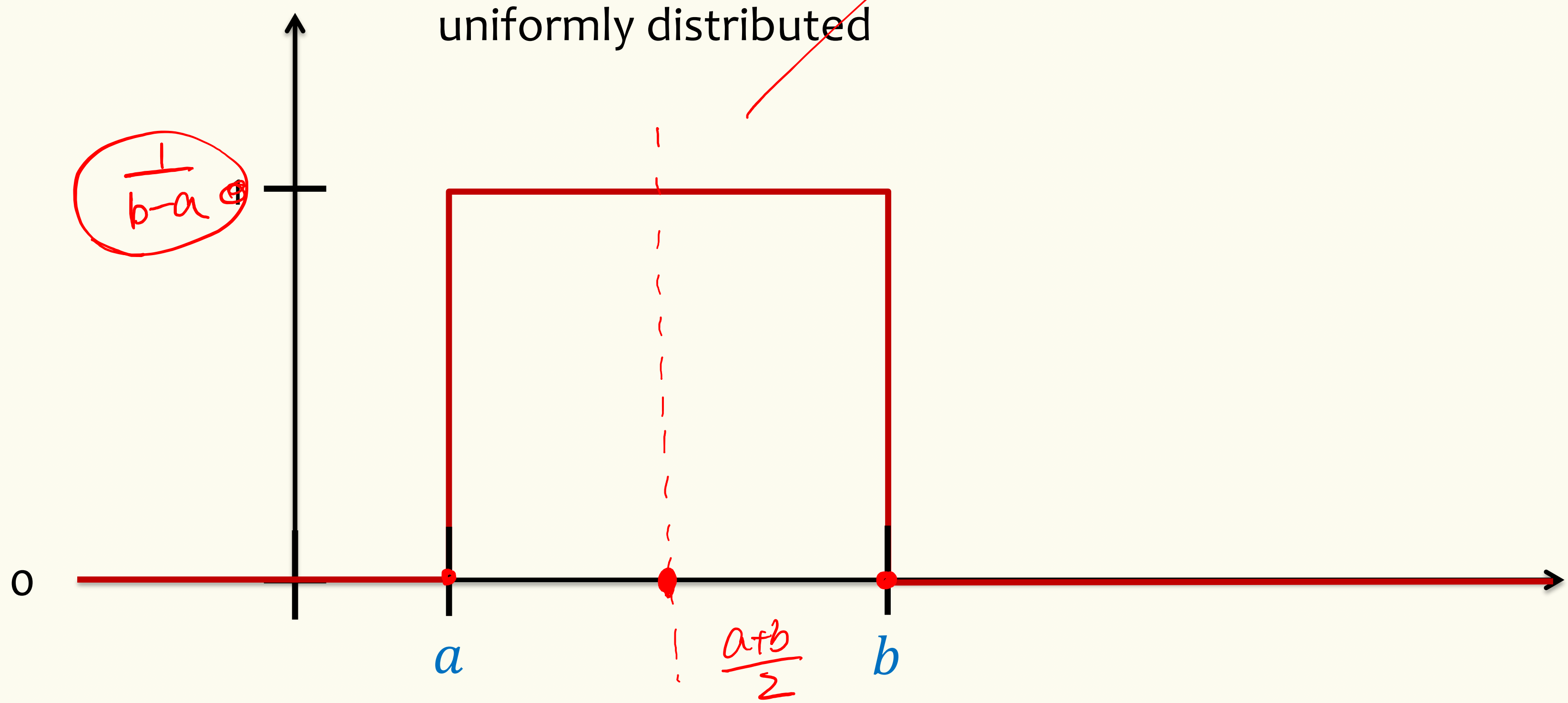
$$\mathbb{E}(T) = \underbrace{\frac{1}{2} 1^2 = \frac{1}{2}}_{\text{Area of triangle}}$$

Uniform Distribution

$X \sim \mathrm{Unif}(a, b)$

We also say that X follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



Uniform Density – Expectation

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \mathrm{Unif}(a,b)$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

$$= \frac{1}{b-a} \int_a^b x \, \mathrm{d}x = \frac{1}{b-a} \left(\frac{x^2}{2} \right) \Bigg|_a^b = \frac{1}{b-a} \left(\frac{b^2 - a^2}{2} \right)$$

$$= \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2}$$

Uniform Density – Variance

$X \sim \text{Unif}(a, b)$

$Var(X) = E(X^2) - E(X)^2$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, \mathrm{d}x$$

$$= \frac{1}{b-a} \int_a^b x^2 \, \mathrm{d}x = \frac{1}{b-a} \boxed{\left(\frac{x^3}{3}\right)} \Bigg|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

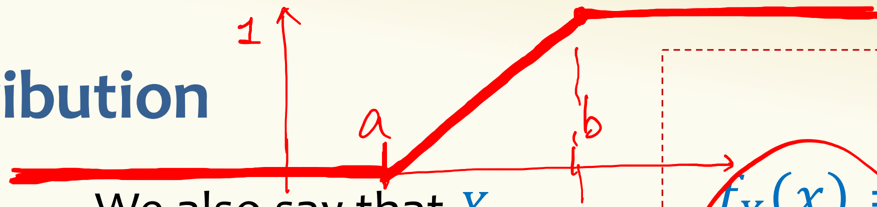
Uniform Density – Variance

$$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}(X) = \frac{a+b}{2}$$

$X \sim \mathrm{Unif}(a, b)$

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 \\
&= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4} \\
&= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12} \\
&= \frac{b^2 - 2ab + a^2}{12} = \boxed{\frac{(b-a)^2}{12}}
\end{aligned}$$

Uniform Distribution



$X \sim \mathrm{Unif}(a, b)$

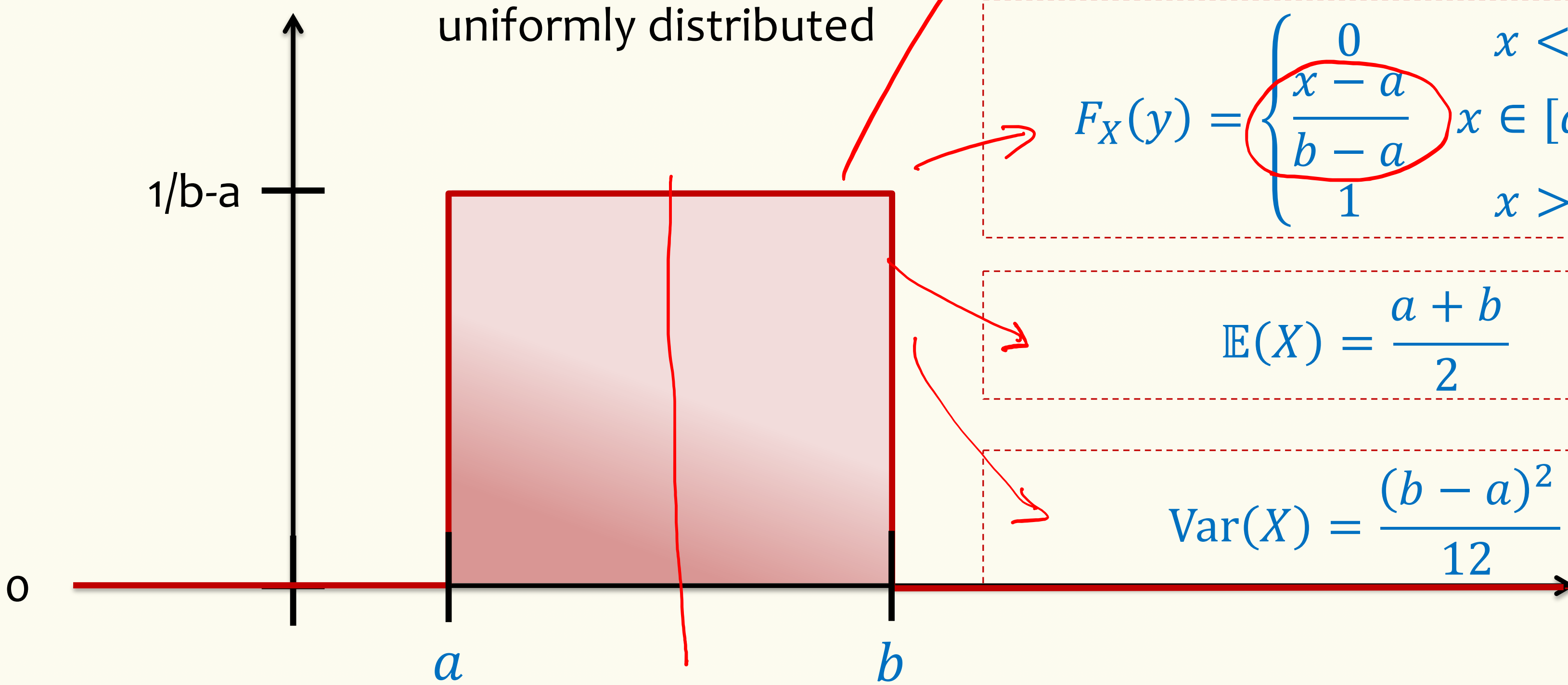
We also say that X follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$F_X(y) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & x \in [a, b] \\ 1 & x > b \end{cases}$$

$$\mathbb{E}(X) = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$



Exponential Density

$\lambda = 10$

Assume expected # of occurrences of an event per unit of time is λ

- Cars going through intersection
- Number of lightning strikes
- Requests to web server
- Patients admitted to ER

$\frac{1}{10}$ T

0 # 1

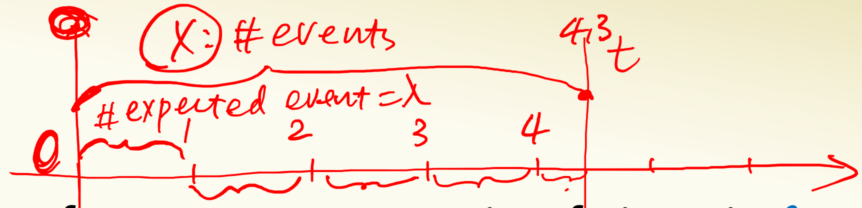
Numbers of occurrences of event: Poisson distribution

$$\mathbb{P}(X = i) = \boxed{e^{-\lambda} \frac{\lambda^i}{i!}} \qquad \text{(Discrete)}$$

How long to wait until ~~next~~ first event? Exponential density!

Let's define it and then derive it!

The Exponential PDF/CDF



Assume expected # of occurrences of an event per unit of time is λ

Numbers of occurrences of event: Poisson distribution

How long to wait until next event? Exponential density!

Y = time of first event

- The exponential RV has range $[0, \infty]$, unlike Poisson with range $\{0,1,2,\dots\}$
- Let $Y \sim Exp(\lambda)$ be the time till the first event. We will compute $F_Y(t)$ and $f_Y(t)$
- Let $X \sim Poi(t\lambda)$ be the # of events in the first t units of time, for $t \geq 0$.
- $P(Y > t) = P(no\ event\ in\ the\ first\ t\ units) = P(X = 0) = e^{-t\lambda}\frac{(t\lambda)^0}{0!} = e^{-t\lambda}$
- $F_Y(t) = 1 - P(Y > t) = 1 - e^{-t\lambda}$
- $f_Y(t) = \frac{d}{dt}F_Y(t) = \lambda e^{-t\lambda}$

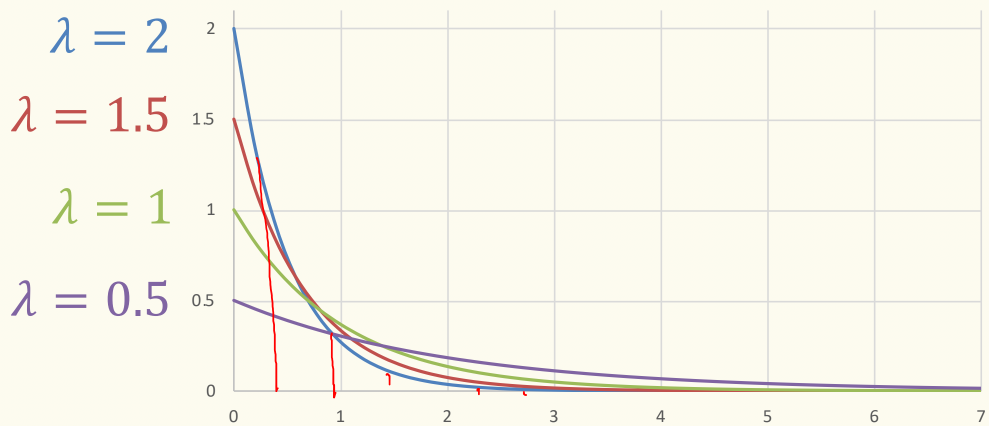
Exponential Distribution

Definition. An **exponential random variable** X with parameter $\lambda \geq 0$ is follows the exponential density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

We write $X \sim \mathrm{Exp}(\lambda)$ and say X that follows the exponential distribution.

CDF: For $y \geq 0$,
$F_X(y) = 1 - e^{-\lambda y}$



Expectation

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

if $\lambda = 10$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

first event heppen at $\frac{1}{10}$

$$= \int_0^{+\infty} \boxed{\lambda e^{-\lambda x}} \cdot x \, \mathrm{d}x$$

if $\lambda = 0.1$

first event heppen at $t = 10$

$$= \left(-(x + \frac{1}{\lambda}) e^{-\lambda x} \right) \Big|_0^{\infty} = \frac{1}{\lambda}$$

$$\mathbb{E}(X) = \frac{1}{\lambda}$$

$$\mathrm{Var}(X) = \frac{1}{\lambda^2}$$

Somewhat complex calculation
use integral by parts

Memorylessness

Definition. A random variable is **memoryless** if for all $s, t > 0$,

$$\mathbb{P}(X > s + t \mid X > s) = \boxed{\mathbb{P}(X > t)}.$$

P(first event after s+t
given, first event after s)

= P(first event happen after t.)

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

s=10

Assuming exp distr, if you've waited s minutes,
prob of waiting t more is exactly same as $s = 0$

Memorylessness of Exponential

Assuming exp distr, if you've waited s minutes, prob of waiting t more is exactly same as $s = 0$

Fact. $X \sim \text{Exp}(\lambda)$ is memoryless.

Proof.

$$\mathbb{P}(X > s + t \mid X > s) = \frac{\mathbb{P}(\{X > s + t\} \cap \{X > s\})}{\mathbb{P}(X > s)}$$

$$= \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)}$$

$$= \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t)$$

The only memoryless RVs are the geometric RV (discrete) and Exp RV (continuous)

example

- Time it takes to check someone out at a grocery store is exponential with an expected value of 10 mins. $\lambda = \frac{1}{10}$
- Independent for different customers
- If you are the second person in line, what is the probability that you will have to wait between 10 and 20 mins.

$$T \sim exp(10^{-1})$$

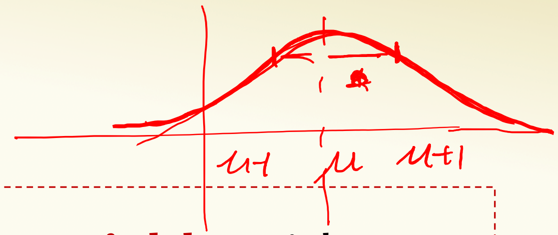
$$Pr(10 \leq T \leq 20) = \int_{10}^{20} \frac{1}{10} e^{-\frac{x}{10}} dx$$

$dx = dy \times 10$

$$y = \frac{x}{10} \qquad dy = \frac{1}{10} dx$$

$$Pr(10 \leq T \leq 20) = \int_{1}^{2} e^{-y} dy = -\left. e^{-y} \right|_{1}^{2} = (e^{-1} - e^{-2})$$

The Normal Distribution



Definition. A **Gaussian (or normal) random variable** with parameters $\mu \in \mathbb{R}$ and $\sigma \geq 0$ has density

expectation   standard deviation

$$f_X(x) = \boxed{\frac{1}{\sqrt{2\pi}\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

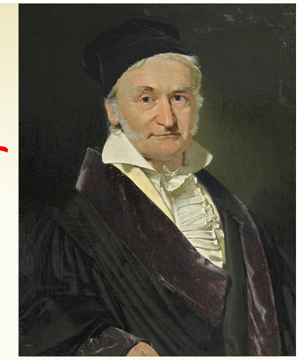
$$\int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sqrt{2\pi}\cdot\sigma$$

(We say that X follows the Normal Distribution, and write $X \sim \mathcal{N}(\mu, \sigma^2)$)

Fact. If $X \sim \mathcal{N}(\mu, \sigma^2)$, then $\mathbb{E}(X) = \mu$, and $\mathrm{Var}(X) = \sigma^2$

Proof is easy because density curve is symmetric around μ, $f_X(\mu - x) = f_X(\mu + x)$

We will see next time why the normal distribution is (in some sense) the most important distribution.



Carl Friedrich Gauss

The Normal Distribution

Aka a "Bell Curve" (imprecise name)