CSE 312

Foundations of Computing II

Lecture 14: Continuous RV

PAUL G. ALLEN SCHOOL OF COMPUTER SCIENCE & ENGINEERING

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Agenda

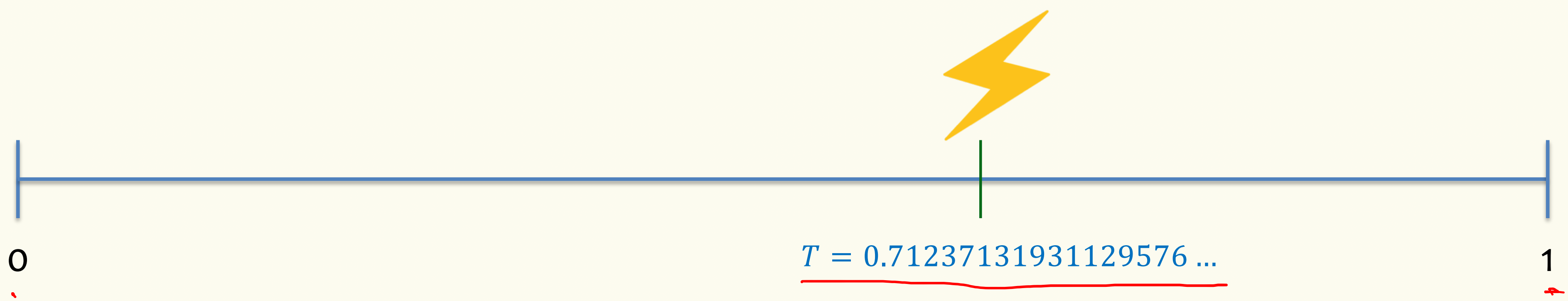
- Continuous Random Variables
- Probability Density Function
- Cumulative Distribution Function

Often we want to model experiments where the outcome is not discrete.

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
  - Time measured with infinitesimal precision.

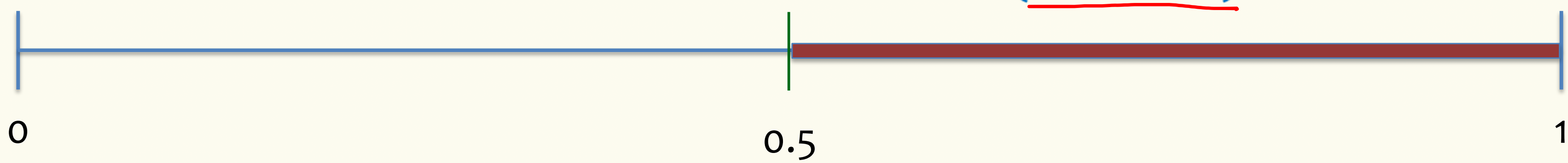


The outcome space is not discrete

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within [0,1] is equally likely

$\mathbb{P}(T \geq 0.5) = ½$



Lightning strikes a pole within a one-minute time frame

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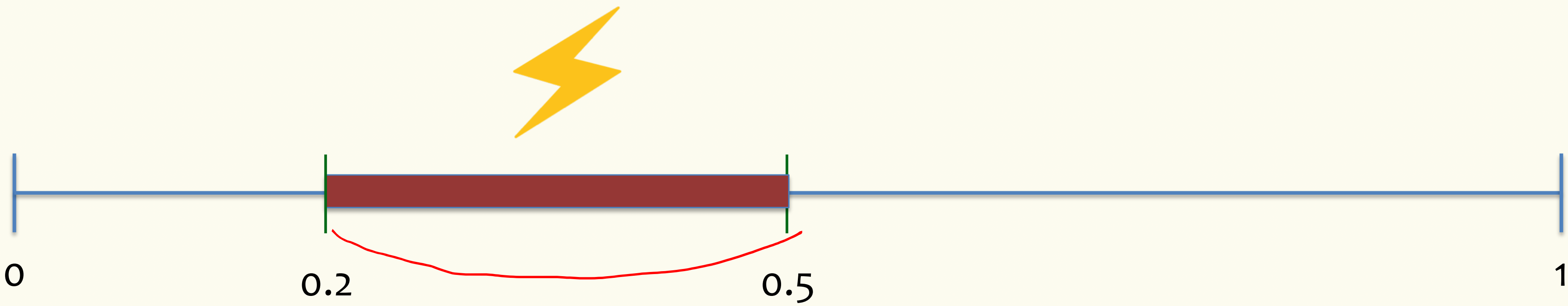
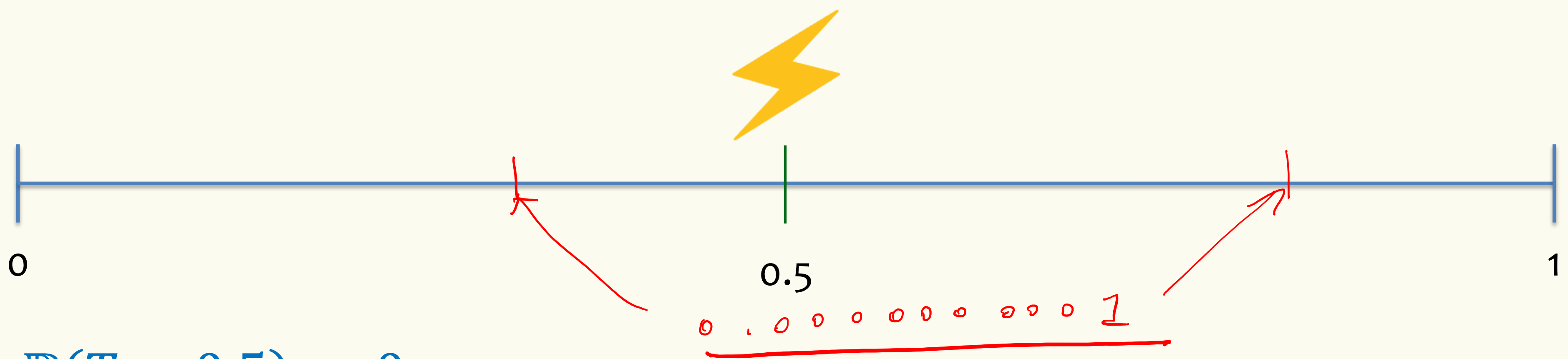


$$\mathbb{P}(0.2 \leq T \leq 0.5) = 0.5 - 0.2 = 0.3$$

Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within [0,1] is equally likely



$\mathbb{P}(T = 0.5) = 0$

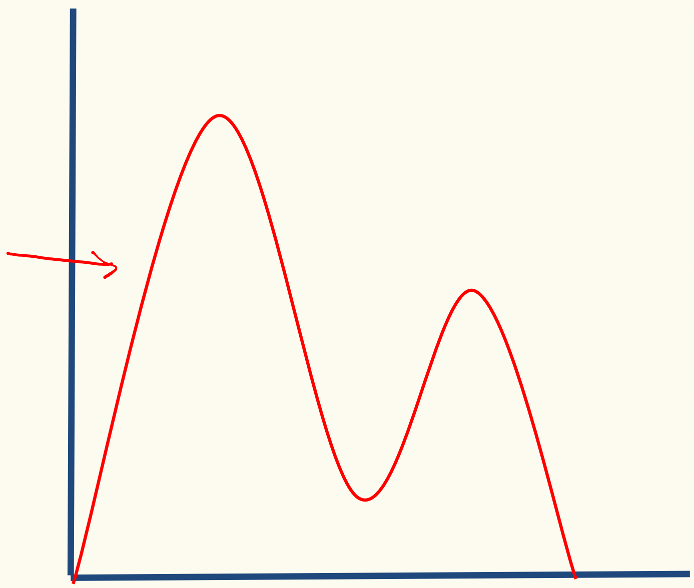
Bottom line

- This gives rise to a different type of random variable
- $\mathbb{P}(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
  - $\mathbb{P}(T \in [0,1]) = 1$
  - $\mathbb{P}(T \in [a, b]) = b - a$
  - …
- How do we model the behavior of T?

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \to \mathbb{R}$, such that

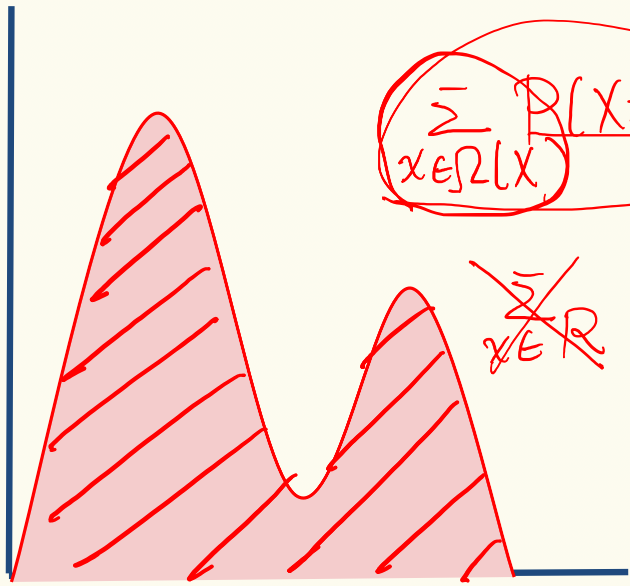
MASS

$f_X: x \to P(X=x)$



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

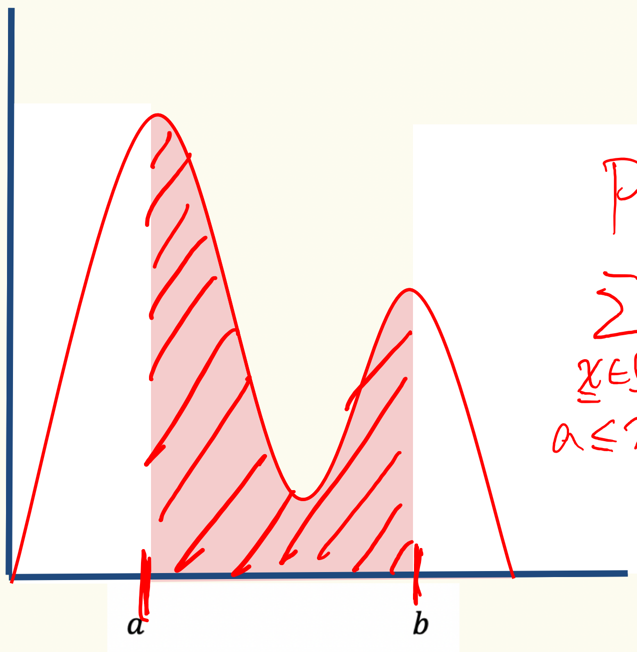
Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = 1$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

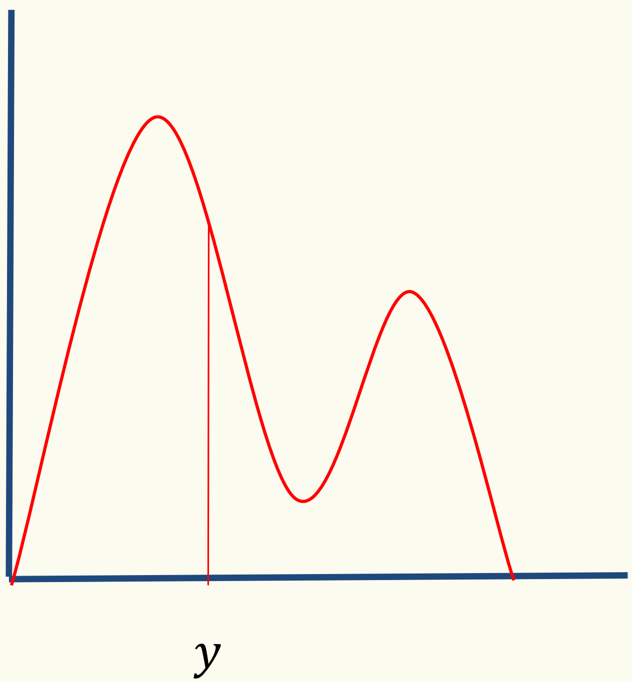
$P(a \leq X \leq b)$

$\sum_{\substack{x \in \Omega(X) \\ a \leq x \leq b}} P(X = x)$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$P(a < X < b)$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

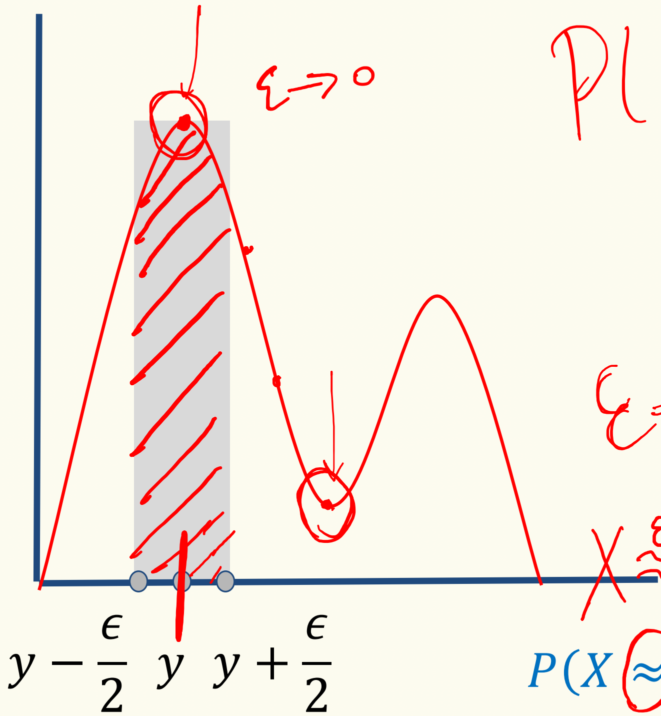
$\mathbb{P}(a < X < b) \approx$ $P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$

$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$

Density ≠ Probability

$f_X(y) \neq 0$ $\qquad$ $\mathbb{P}(X = y) = 0$

Probability Density Function - Intuition



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

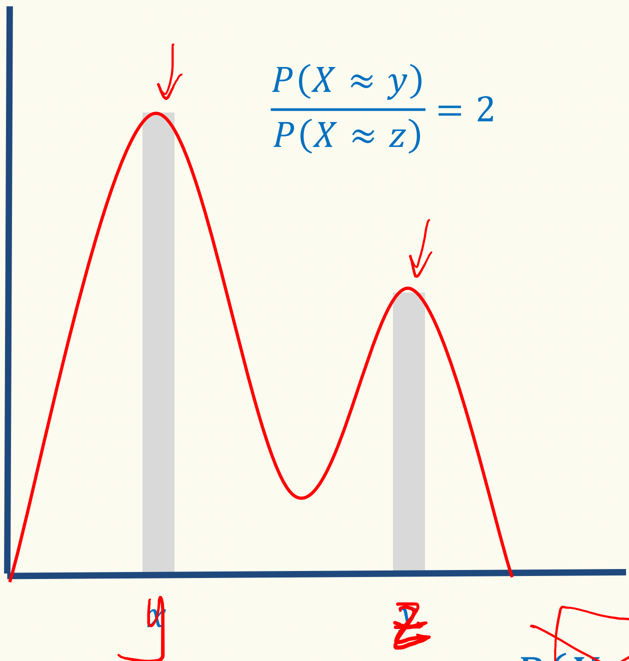
Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

Probability Density Function - Intuition



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$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

Definition. A **continuous random variable** X is defined by a **probability density function** (PDF) $f_X: \mathbb{R} \to \mathbb{R}$, such that

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

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$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

$$P(X \approx y) \approx P\left(y - \frac{\epsilon}{2} \leq X \leq y + \frac{\epsilon}{2}\right) = \int_{y-\frac{\epsilon}{2}}^{y+\frac{\epsilon}{2}} f_X(x)\,\mathrm{d}x \approx \epsilon f_X(y)$$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)}$$

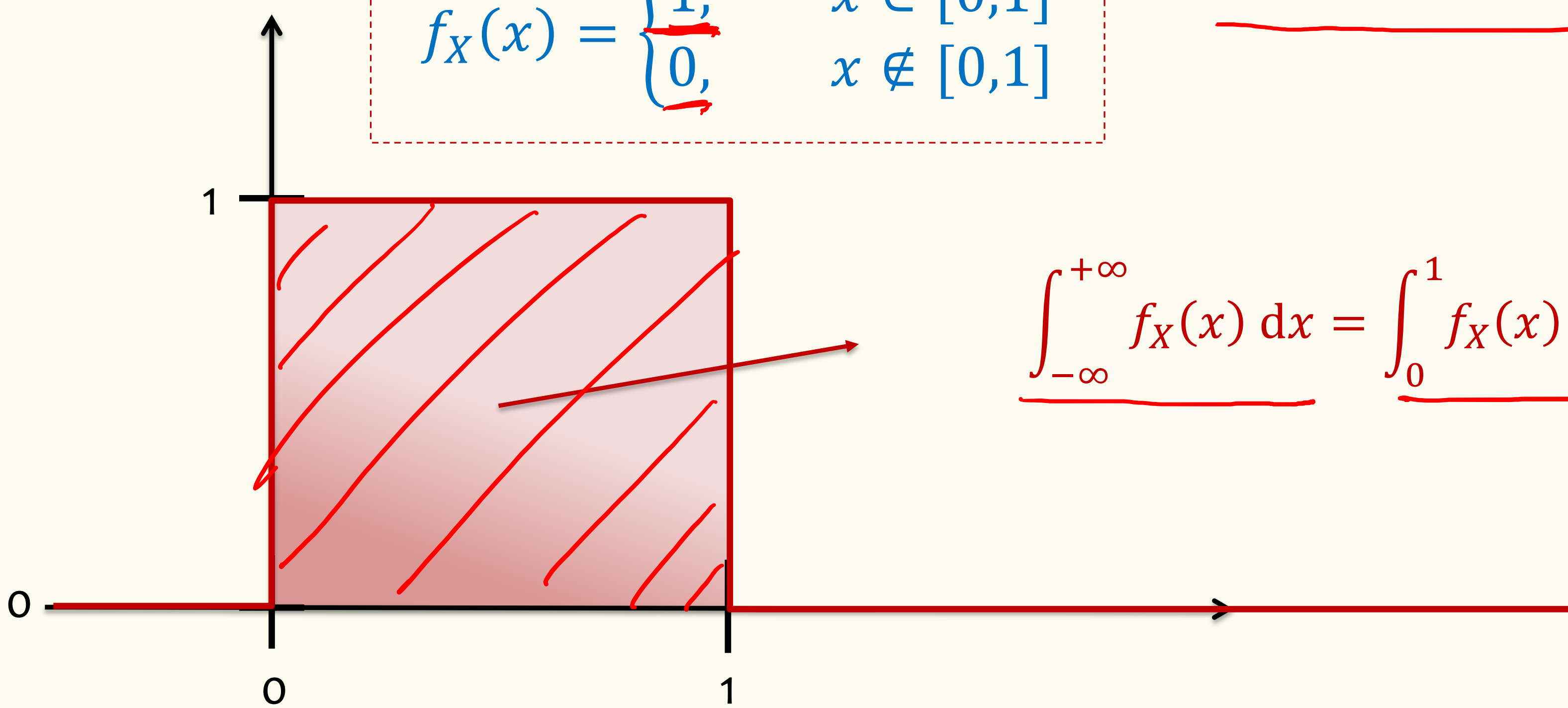
PDF of Uniform RV

$X \sim \text{Unif}(0,1)$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

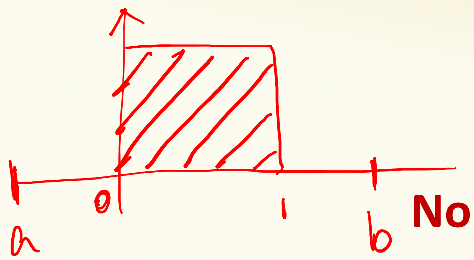
Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

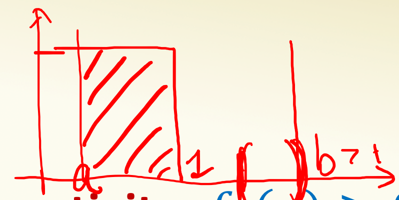
Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$



$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_{0}^{1} f_X(x)\,\mathrm{d}x = 1 \cdot 1 = 1$$

Probability of Event





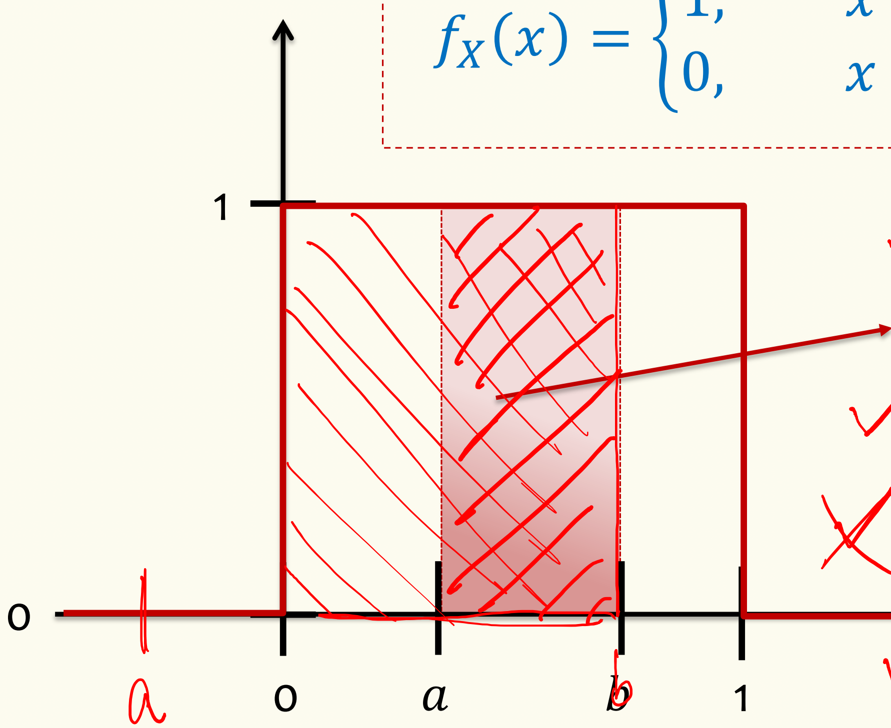
$X \sim \text{Unif}(0,1)$

Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\, dx = 1$

$P(a \leq X \leq b) = \int_a^b f_X(x)\, dx$ ~~$= 1$~~



1. If $0 \leq a \; and \; b \leq 1$
   $\mathbb{P}(a \leq X \leq b) = b - a$
2. If $a < 0 \; and \; b \leq 1$
   $\mathbb{P}(a \leq X \leq b) = b$
3. If $a \geq 0 \; and \; b > 1$
   $\mathbb{P}(a \leq X \leq b) = b - a$
4. If $a < 0 \; and \; b > 1$
   $\mathbb{P}(a \leq X \leq b) = 1$

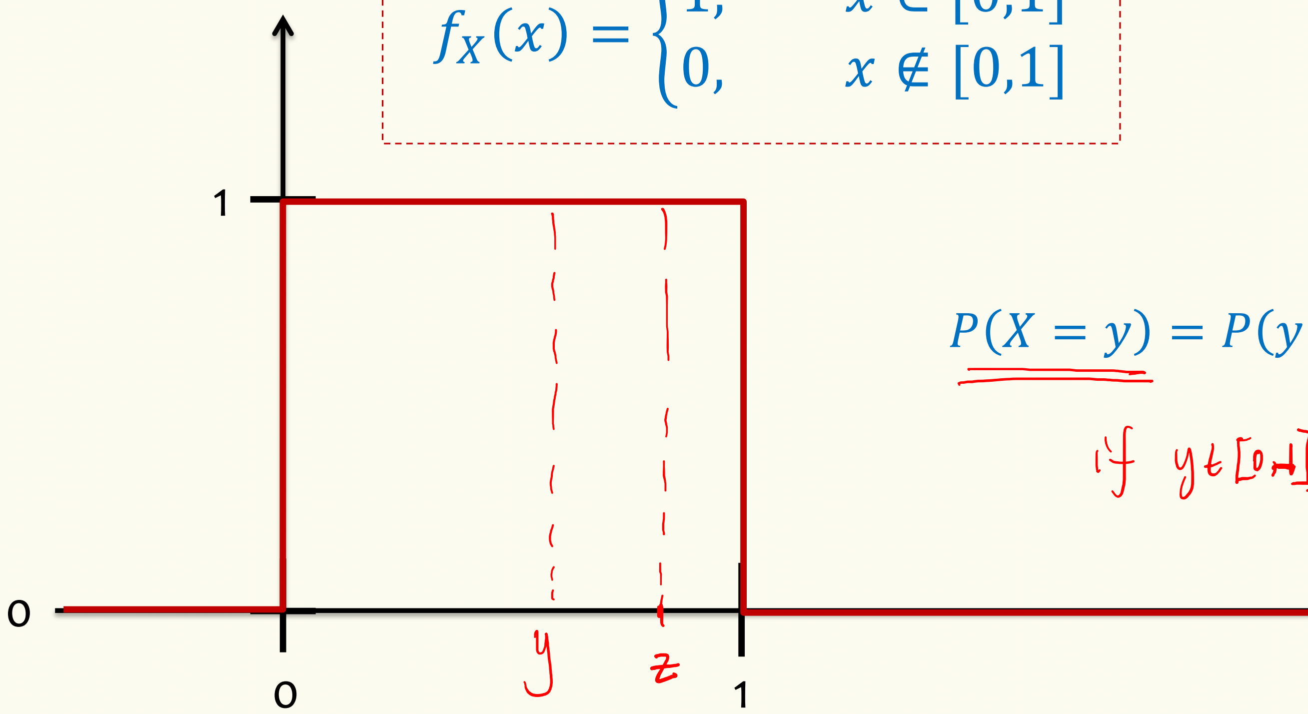
Poll: pollev/rachel312

A. All of them are correct
B. Only 1, 2, 4 are right
C. Only 1 is right
D. Only 1 and 2 are right

Probability of Event

$X \sim \text{Unif}(0,1)$

$$f_X(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



Non-negativity: $f_X(x) \geq 0$ for all $x \in \mathbb{R}$

Normalization: $\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = 1$

$$P(a \leq X \leq b) = \int_a^b f_X(x)\,\mathrm{d}x$$

$$P(X = y) = P(y \leq X \leq y) = \int_y^y f_X(x)\,\mathrm{d}x = 0$$

if $y \in [0,1]$, $P(X \approx y) \approx \epsilon f_X(y) = \epsilon$

$$\frac{P(X \approx y)}{P(X \approx z)} \approx \frac{\epsilon f_X(y)}{\epsilon f_X(z)} = \frac{f_X(y)}{f_X(z)} = 1$$

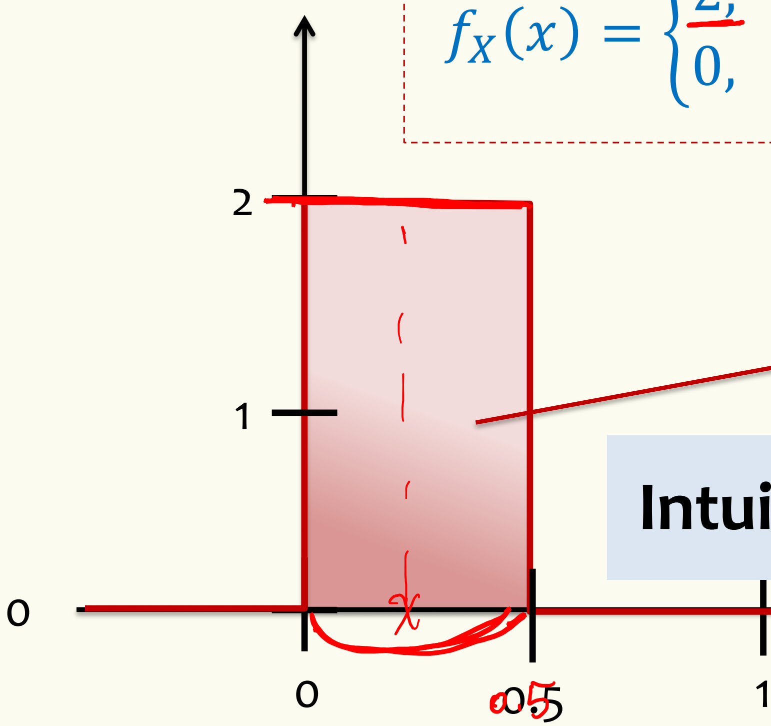
PDF of Uniform RV

Density ≠ Probability

$f_X(x) \gg 1$ is possible!

$X \sim \text{Unif}(0{,}0.5)$

$$f_X(x) = \begin{cases} 2, & x \in [0{,}0.5] \\ 0, & x \notin [0{,}0.5] \end{cases}$$



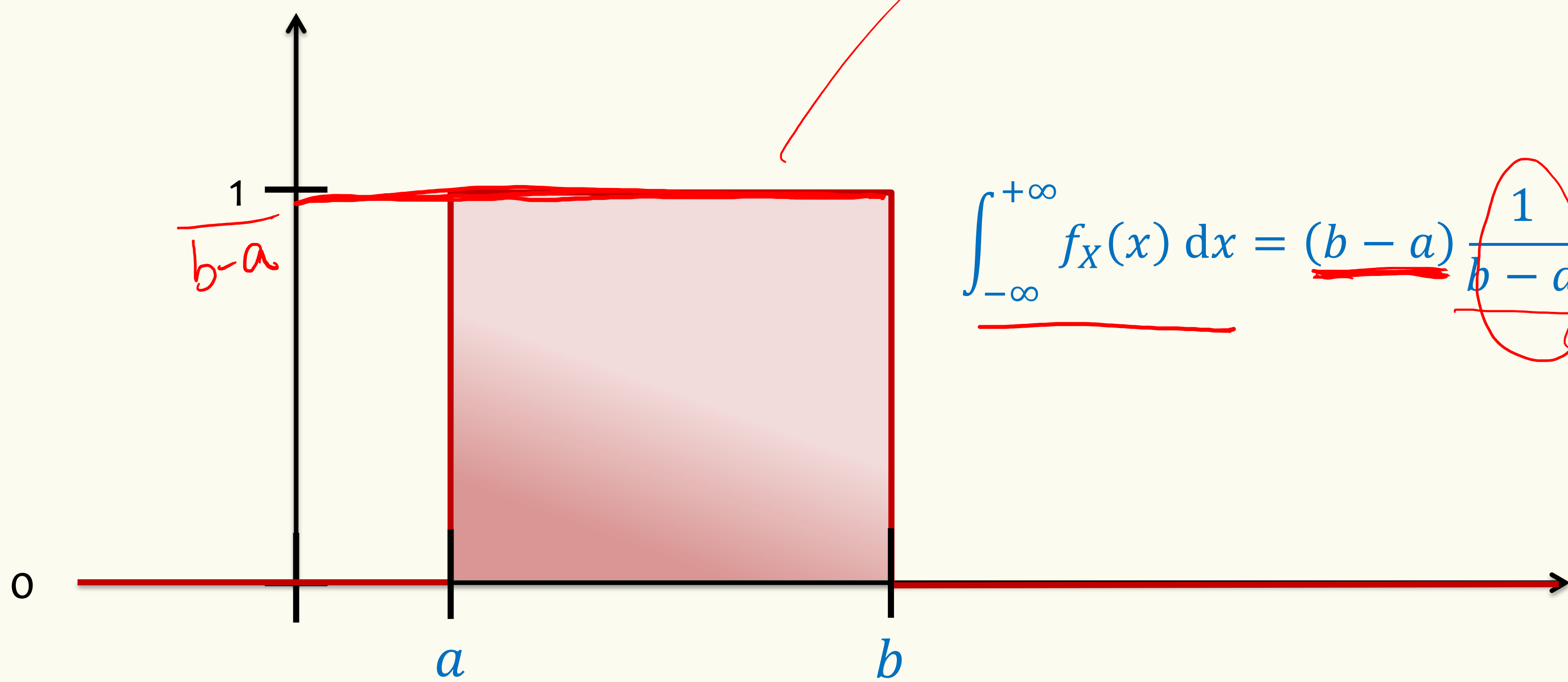
$$\int_{-\infty}^{+\infty} f_X(x)\,\mathrm{d}x = \int_0^1 f_X(x)\,\mathrm{d}x = 2 \cdot 0.5 = 1$$

Intuition: $\mathbb{P}(X \approx x) \approx f_X(x) \cdot \epsilon$ for small ϵ

Uniform Distribution

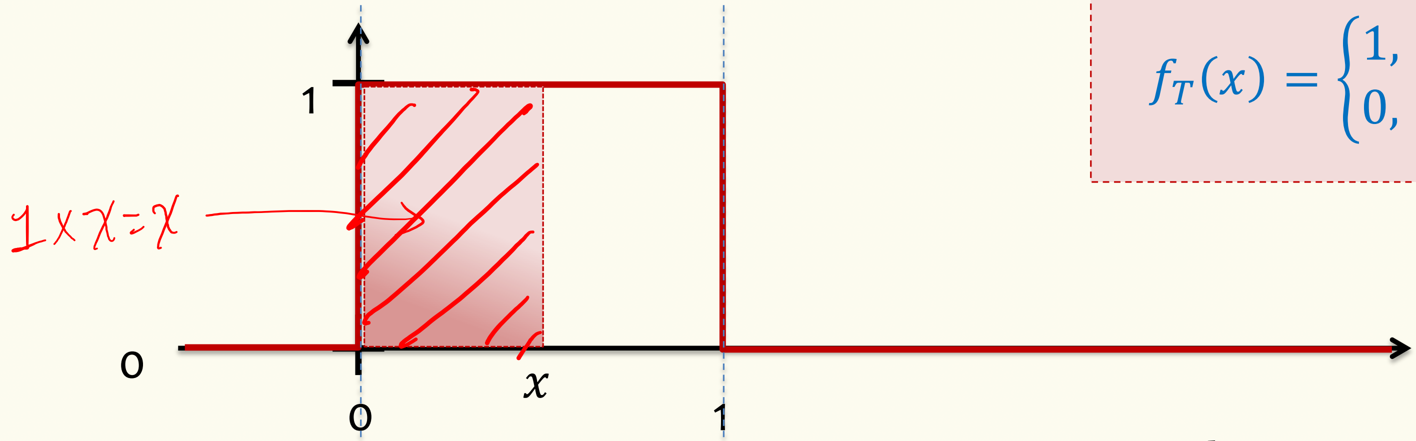
$X \sim \text{Unif}(a, b)$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



$$\int_{-\infty}^{+\infty} f_X(x)\, \mathrm{d}x = (b-a)\frac{1}{b-a} = 1$$

Example. $T \sim \text{Unif}(0,1)$



Probability Density Function

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$

Cumulative Distribution Function

$$F_T(x) = P(T \le x) = \begin{cases} 0 & x \le 0 \\ ? & 0 \le x \le 1 \\ 1 & 1 \le x \end{cases}$$

Cumulative Distribution Function

Definition. The **cumulative distribution function (cdf)** of X is

$$F_X(a) = \mathbb{P}(X \le a) = \int_{-\infty}^{a} f_X(x)\, \mathrm{d}x$$

By the fundamental theorem of Calculus $f_X(x) = \frac{d}{dx}F(x)$

Therefore: $\mathbb{P}(X \in [a,b]) = F(b) - F(a)$

F_X is monotone increasing, since $f_X(x) \ge 0$. That is $F_X(c) \le F_X(d)$ for $c \le d$

$\mathrm{Lim}_{a \to -\infty} F_X(a) = P(X \le -\infty) = 0$ $\quad$ $\mathrm{Lim}_{a \to +\infty} F_X(a) = P(X \le +\infty) = 1$

From Discrete to Continuous

$P(a \le X \le b)$

	Discrete	Continuous
PMF/PDF	$p_X(x) = P(X = x)$	$f_X(x) \neq P(X = x) = 0$
CDF	$F_X(x) = \sum_{t \le x} p_X(t)$	$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt$
Normalization	$\sum_x p_X(x) = 1$	$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$
Expectation	$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$	$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$

Expectation of a Continuous RV

Definition. The **expected value** of a continuous RV X is defined as

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Fact. $\mathbb{E}(aX + bY + c) = a\mathbb{E}(X) + b\mathbb{E}(Y) + c$

Definition. The **variance** of a continuous RV X is defined as

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot \left(x - \mathbb{E}(X)\right)^2 \mathrm{d}x = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

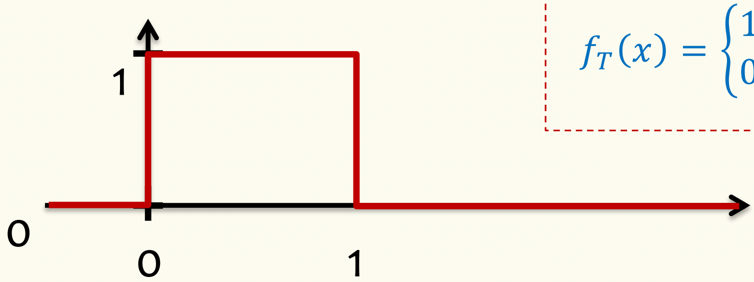
Expectation of a Continuous RV

Definition.

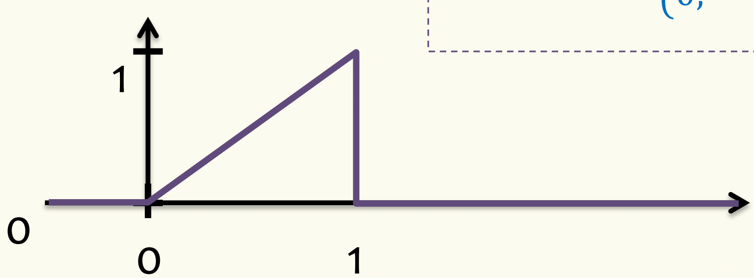
$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \, \mathrm{d}x$$

Example. $T \sim \mathrm{Unif}(0,1)$

$$f_T(x) = \begin{cases} 1, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$f_T(x) \cdot x = \begin{cases} x, & x \in [0,1] \\ 0, & x \notin [0,1] \end{cases}$$



$$\mathbb{E}(T) = \underbrace{\frac{1}{2} 1^2}_{\text{Area of triangle}} = \frac{1}{2}$$

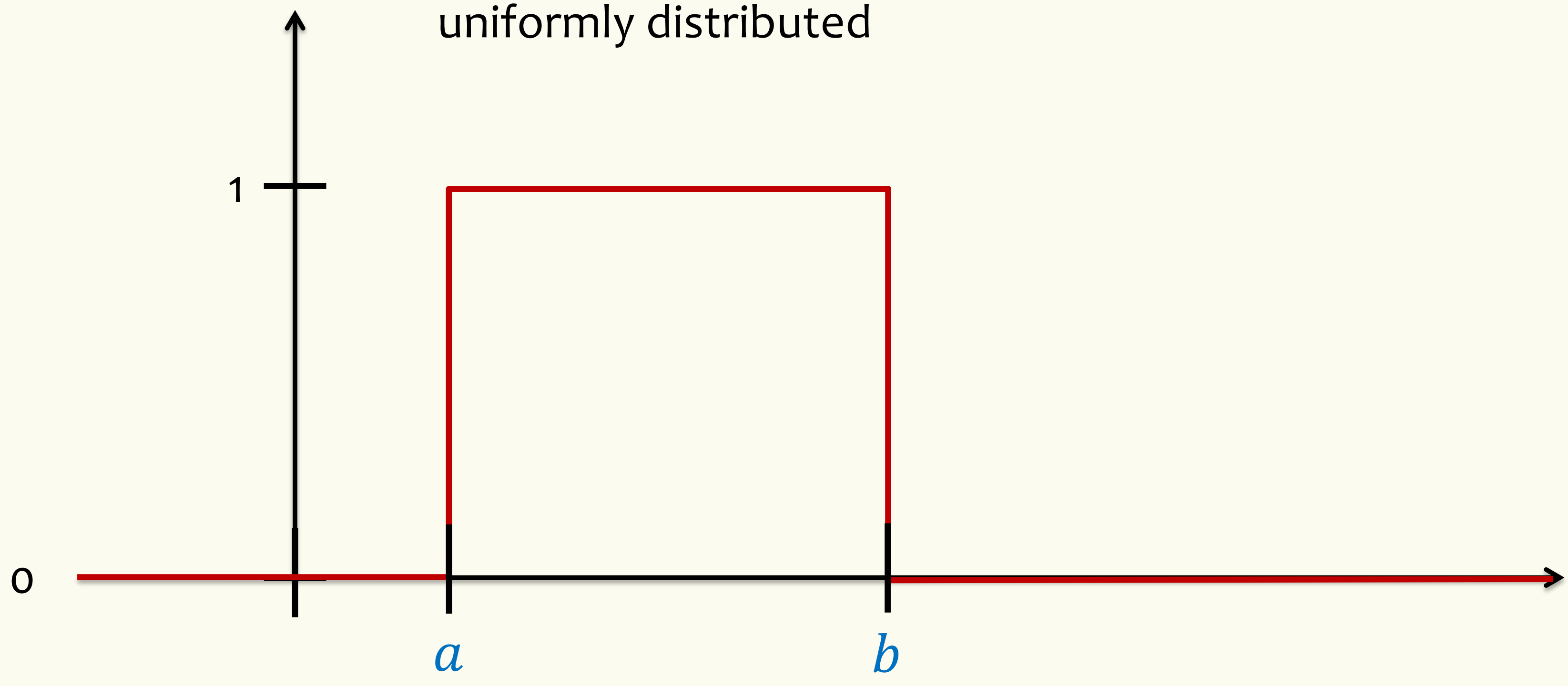
Area of triangle

Uniform Distribution

$X \sim \text{Unif}(a, b)$

We also say that X follows the uniform distribution / is uniformly distributed

$$f_X(x) = \begin{cases} \frac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$



Uniform Density – Expectation

$X \sim \mathrm{Unif}(a, b)$

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a, b] \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E}(X) = \int_{-\infty}^{+\infty} f_X(x) \cdot x \,\mathrm{d}x$$

$$= \frac{1}{b-a}\int_a^b x \,\mathrm{d}x \quad = \frac{1}{b-a}\left(\frac{x^2}{2}\right)\Bigg|_a^b = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right)$$

$$= \frac{(b-a)(a+b)}{2(b-a)} = \frac{a+b}{2}$$

Uniform Density – Variance

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & x \in [a,b] \\ 0 & \text{else} \end{cases}$$

$X \sim \text{Unif}(a,b)$

$$\mathbb{E}(X^2) = \int_{-\infty}^{+\infty} f_X(x) \cdot x^2 \, \mathrm{d}x$$

$$= \frac{1}{b-a} \int_a^b x^2 \, \mathrm{d}x = \frac{1}{b-a} \left(\frac{x^3}{3} \right) \Bigg|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

$$= \frac{(b-a)(b^2 + ab + a^2)}{3(b-a)} = \frac{b^2 + ab + a^2}{3}$$

Uniform Density – Variance

$$\mathbb{E}(X^2) = \frac{b^2 + ab + a^2}{3} \qquad \mathbb{E}(X) = \frac{a+b}{2}$$

$X \sim \text{Unif}(a, b)$

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$= \frac{b^2 + ab + a^2}{3} - \frac{a^2 + 2ab + b^2}{4}$$

$$= \frac{4b^2 + 4ab + 4a^2}{12} - \frac{3a^2 + 6ab + 3b^2}{12}$$

$$= \frac{b^2 - 2ab + a^2}{12} = \frac{(b-a)^2}{12}$$