CSE 312

Foundations of Computing II

Lecture 13: Poisson Distribution

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incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Zoo of Discrete RVs! 🦍🐘🦁🐅🦓🐪🦒

$X \sim \text{Unif}(a, b)$

$$P(X = k) = \frac{1}{b - a + 1}$$

$$E[X] = \frac{a + b}{2}$$

$$Var(X) = \frac{(b - a)(b - a + 2)}{12}$$

$X \sim \text{Ber}(p)$

$$P(X = 1) = p, P(X = 0) = 1 - p$$

$$E[X] = p$$

$$Var(X) = p(1 - p)$$

$X \sim \text{Bin}(n, p)$

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}$$

$$E[X] = np$$

$$Var(X) = np(1 - p)$$

$X \sim \text{Geo}(p)$

$$P(X = k) = (1 - p)^{k-1} p$$

$$E[X] = \frac{1}{p}$$

$$Var(X) = \frac{1 - p}{p^2}$$

$X \sim \text{NegBin}(r, p)$

$$P(X = k) = \binom{k - 1}{r - 1} p^r (1 - p)^{k-r}$$

$$E[X] = \frac{r}{p}$$

$$Var(X) = \frac{r(1 - p)}{p^2}$$

$X \sim \text{HypGeo}(N, K, n)$

$$P(X = k) = \frac{\binom{K}{k}\binom{N-K}{n-k}}{\binom{N}{n}}$$

$$E[X] = n\frac{K}{N}$$

$$Var(X) = n\frac{K(N - K)(N - n)}{N^2(N - 1)}$$

Agenda

- Poisson Distribution
- Approximate Binomial distribution using Poisson distribution

Poisson Distribution

Siméon Denis Poisson
1781-1840

- Suppose "events" happen, independently, at an *average* rate of λ per unit time.
- Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. *with parameter* λ (denoted X ~ Poi(λ)) and has distribution (PMF):

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

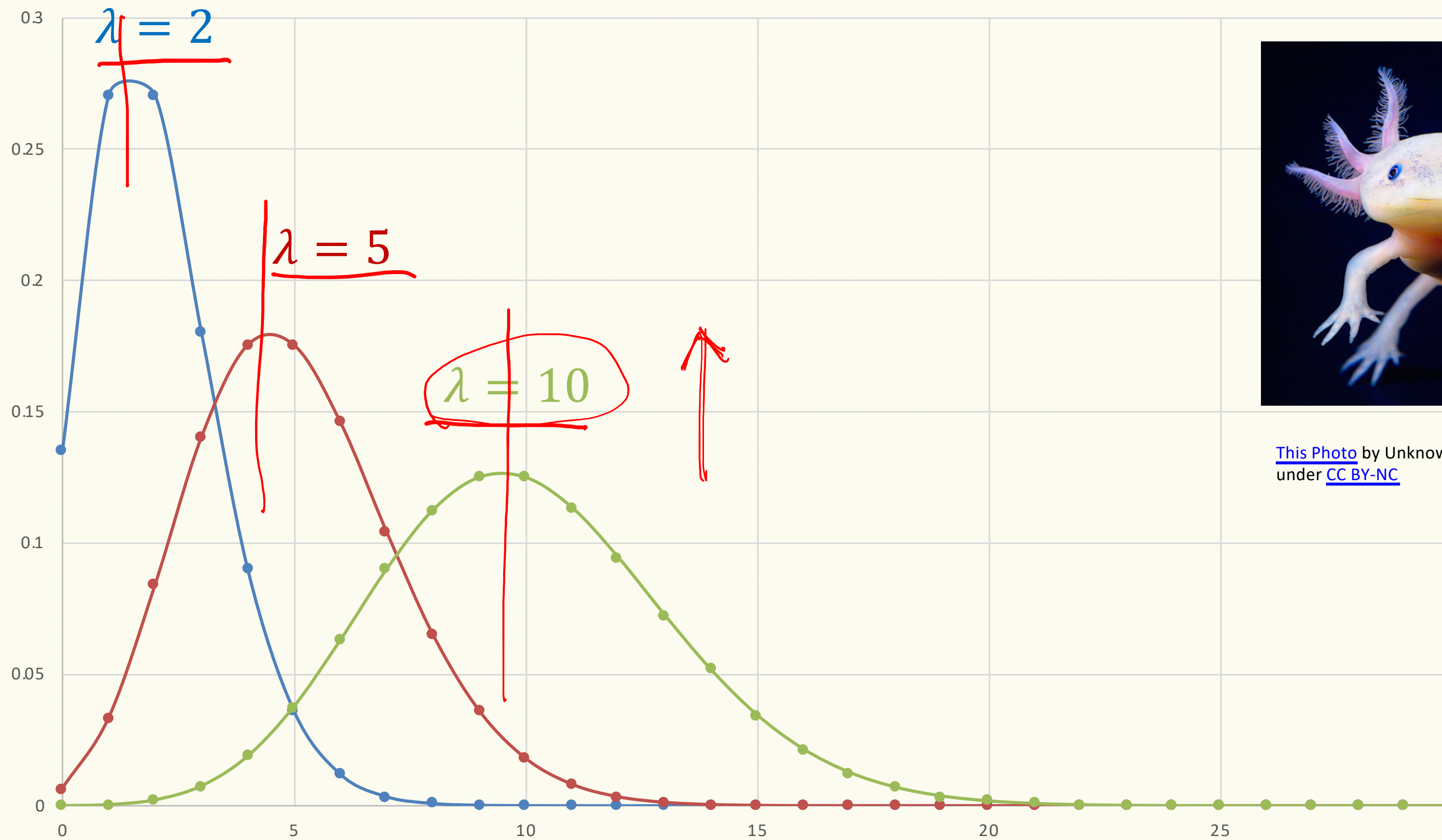
Several examples of "Poisson processes":

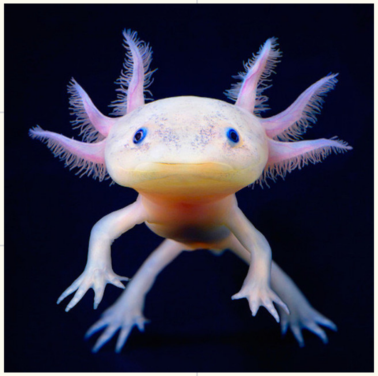
- # of cars passing through a traffic light in 1 hour
- # of requests to web servers in an hour
- # of photons hitting a light detector in a given interval
- # of patients arriving to ER within an hour

Assume fixed average rate

Probability Mass Function

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$





Validity of Distribution

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

We first want to verify that Poisson probabilities sum up to 1.

$$\sum_{i=0}^{\infty} \mathbb{P}(X = i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$$

Fact. $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$

Expectation

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then

$$\mathbb{E}(X) = \lambda$$

Proof.

$$\mathbb{E}(X) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!}$$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!}$$

= 1 (see prior slides!)

$$= \lambda \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} = \lambda \cdot 1 = \lambda$$

Variance

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

Theorem. If X is a Poisson RV with parameter λ, then $\mathrm{Var}(X) = \lambda$

Proof. $\mathbb{E}(X^2) = \sum_{i=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{i!} \cdot i^2 = \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^i}{(i-1)!} i$

$$= \lambda \sum_{i=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^{i-1}}{(i-1)!} \cdot i \quad = \lambda \sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot (j+1)$$

$$= \lambda \left[\underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!} \cdot j}_{= \mathbb{E}(X) = \lambda} + \underbrace{\sum_{j=0}^{\infty} e^{-\lambda} \cdot \frac{\lambda^j}{j!}}_{= 1} \right] = \lambda^2 + \lambda$$

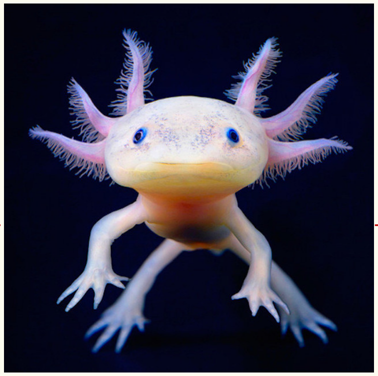
Similar to the previous proof
Verify offline.

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$



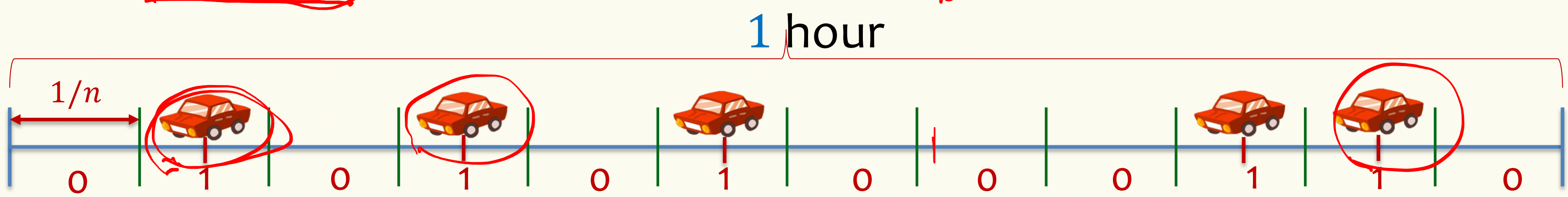
Poisson approximates binomial when n is very large, p is very small, and λ = np is "moderate" (e.g. $n > 20$ and $p < 0.05$, $n > 100$ and $p < 0.1$)

Formally, Binomial is Poisson in the limit as $n \to \infty$ (equivalently, $p \to 0$) while holding $np = \lambda$

Example – Model the process of cars passing through a light in 1 hour

X = # cars passing through a light in 1 hour

Know: $\mathbb{E}(X) = \lambda$ for some given $\lambda > 0$



Discretize problem: n intervals, each of length $\frac{1}{n}$.

In each interval, a car passes by with probability $\frac{\lambda}{n}$ (assume ≤ 1 car can pass by)

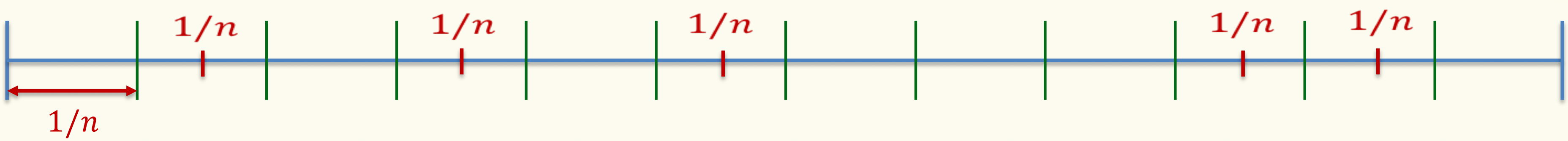
Bernoulli $X_i = 1$ if car in i-th interval (0 otherwise). $\mathbb{P}(X_i = 1) = \frac{\lambda}{n}$

$X = \sum_{i=1}^{n} X_i$ $\quad$ $X \sim$ binomial(n,p) $\quad$ $\mathbb{P}(X = i) = \binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1 - \frac{\lambda}{n}\right)^{n-i}$

indeed! $\mathbb{E}(X) = \lambda$

Don’t like discretization

X is binomial $\mathbb{P}(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}$



We want now $n \to \infty$

$\forall i$

$$\mathbb{P}(X=i)=\binom{n}{i}\left(\frac{\lambda}{n}\right)^i\left(1-\frac{\lambda}{n}\right)^{n-i}=\frac{n!}{(n-i)!\,n^i}\frac{\lambda^i}{i!}\left(1-\frac{\lambda}{n}\right)^n\left(1-\frac{\lambda}{n}\right)^{-i}$$

$\to 1$ $\quad$ $\to e^{-\lambda}$ $\quad$ $\to 1$

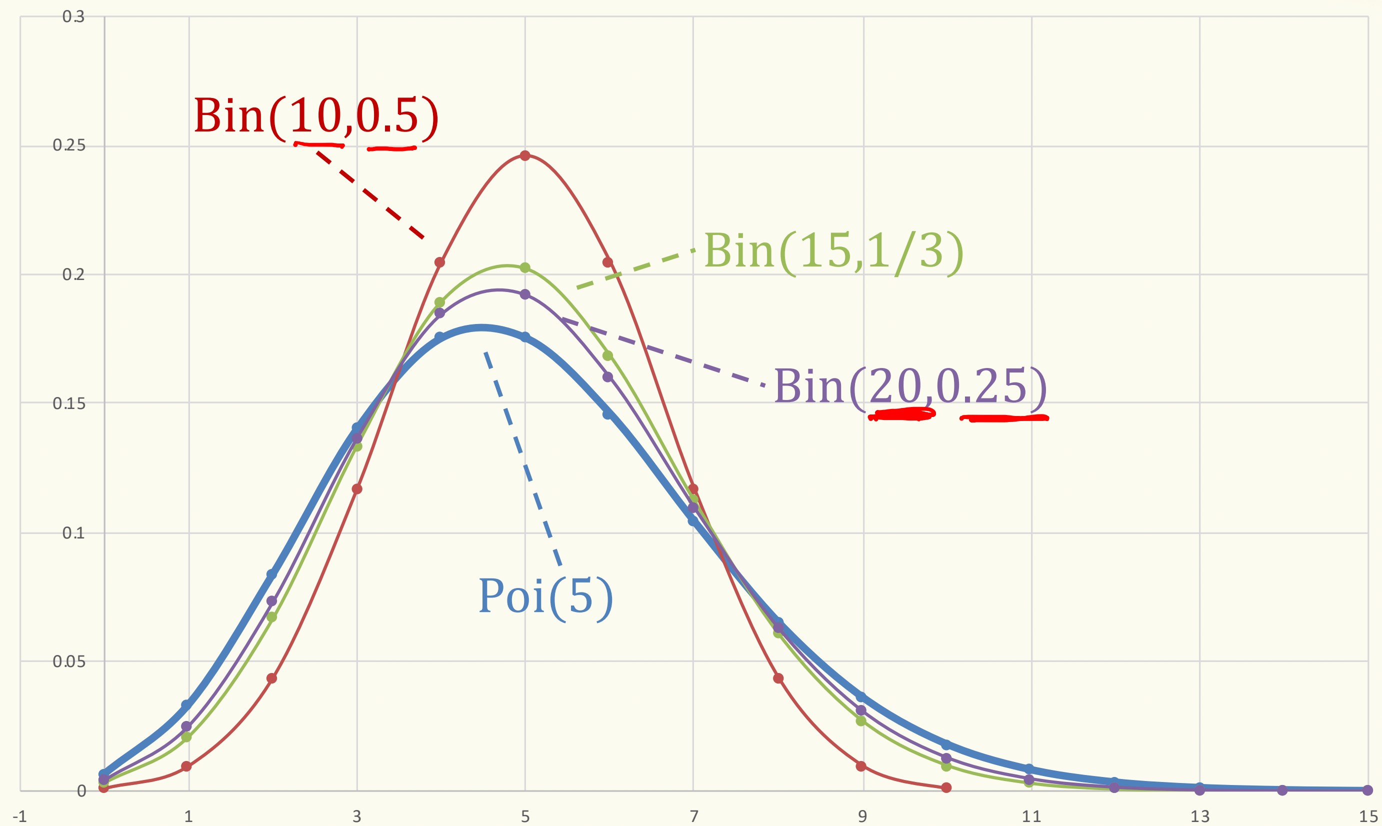
$$\to \mathbb{P}(X=i)=e^{-\lambda}\cdot\frac{\lambda^i}{i!}$$

Probability Mass Function – Convergence of Binomials

$\lambda = 5$

$p = \frac{5}{n}$

$n = 10,15,20$



$as\ n \to \infty,\ \ Binomial(n,\ \ p = \ \lambda/n) \to poi(\lambda)$

From Binomial to Poisson

$X \sim \text{Bin}(n, p)$

$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$

$E[X] = np$

$Var(X) = np(1-p)$

$= np - np^2$

$\lambda - \lambda \cdot p$

$n \to \infty$

$np = \lambda$

$p = \frac{\lambda}{n} \to 0$

$X \sim \text{Poisson}(\lambda)$

$P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$E[X] = \lambda$

$Var(X) = \lambda$

Example -- Approximate Binomial Using Poisson

Consider sending bit string over a network

- Send bit string of length $n = 10^4$
- Probability of (independent) bit corruption is $p = 10^{-6}$
- What is probability that message arrives uncorrupted?

Using $X \sim \text{Poi}(\lambda = np = 10^4 \bullet 10^{-6} = 0.01)$

$$\mathbb{P}(X = 0) = e^{-\lambda} \cdot \frac{\lambda^0}{0!} = e^{-0.01} \cdot \frac{0.01^0}{0!} = 0.990049834$$

Using $Y \sim \text{Bin}(10^4, 10^{-6})$

$$\mathbb{P}(Y = 0) \approx 0.990049829$$

Sum of Independent Poisson RVs

Theorem. Let $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = (X + Y)$. For all $z = 0,1,2,3 \ldots,$

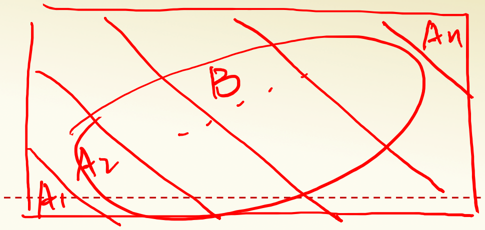
$$\mathbb{P}(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

$E(Z) = E(X) + E(Y)$

More generally, let $X_1 \sim Poi(\lambda_1), \cdots, X_n \sim Poi(\lambda_n)$ such that $\lambda = \Sigma_i \lambda_i$.
Let $Z = \Sigma_i X_i$

$$\mathbb{P}(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

Sum of Independent Poisson RVs



Theorem. Let $X \sim Poi(\lambda_1)$ and $Y \sim Poi(\lambda_2)$ such that $\lambda = \lambda_1 + \lambda_2$.
Let $Z = (X + Y)$. For all $z = 0,1,2,3 \ldots,$

$$\mathbb{P}(Z = z) = e^{-\lambda} \cdot \frac{\lambda^z}{z!}$$

$\mathbb{P}(Z = z) = ?$

1. $\mathbb{P}(Z = z) = \Sigma_{j=0}^{z}\, \mathbb{P}(X = j, Y = z - j)$
2. $\mathbb{P}(Z = z) = \Sigma_{j=0}^{\infty} \mathbb{P}(X = j, Y = z - j)$
3. $\mathbb{P}(Z = z) = \Sigma_{j=0}^{z} \mathbb{P}(Y = z - j | X = j) \mathbb{P}(X = j)$
4. $\mathbb{P}(Z = z) = \Sigma_{j=0}^{z} \mathbb{P}(Y = z - j | X = j)$

Poll: pollev/rachel312

A. All of them are right
B. The first 3 are right
C. Only 1 is right
D. Don't know

$$\mathbb{P}(Z=z) = \Sigma_{j=0}^{k} \mathbb{P}(X=j, Y=z-j)$$

Law of total probability

$$= \Sigma_{j=0}^{k} \mathbb{P}(X=j)\mathbb{P}(Y=z-j) = \Sigma_{j=0}^{k}\, e^{-\lambda_1} \cdot \frac{\lambda_1^j}{j!} \cdot e^{-\lambda_2} \cdot \frac{\lambda_2^{z-j}}{z-j!}$$

Independence

$$= e^{-\lambda} \left(\Sigma_{j=0}^{k} \cdot \frac{1}{j!\, z-j!} \cdot \lambda_1^j \lambda_2^{z-j} \right)$$

$$= e^{-\lambda} \left(\Sigma_{j=0}^{k} \frac{z!}{j!\, z-j!} \cdot \lambda_1^j \lambda_2^{z-j} \right) \frac{1}{z!}$$

$$= e^{-\lambda} \cdot (\lambda_1 + \lambda_2)^z \cdot \frac{1}{z!} = e^{-\lambda} \cdot \lambda^z \cdot \frac{1}{z!}$$

Binomial Theorem

$(x+y)^k$

Poisson Random Variables

Definition. A **Poisson random variable** X with parameter $\lambda \geq 0$ is such that for all $i = 0,1,2,3 \ldots,$

$$\mathbb{P}(X = i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

General principle:

- Events happen at an average rate of λ per time unit
- Number of events happening at a time unit X is distributed according to Poi(λ)
- Poisson approximates Binomial when n is large, p is small, and np is moderate
- Sum of independent Poisson is still a Poisson

Next

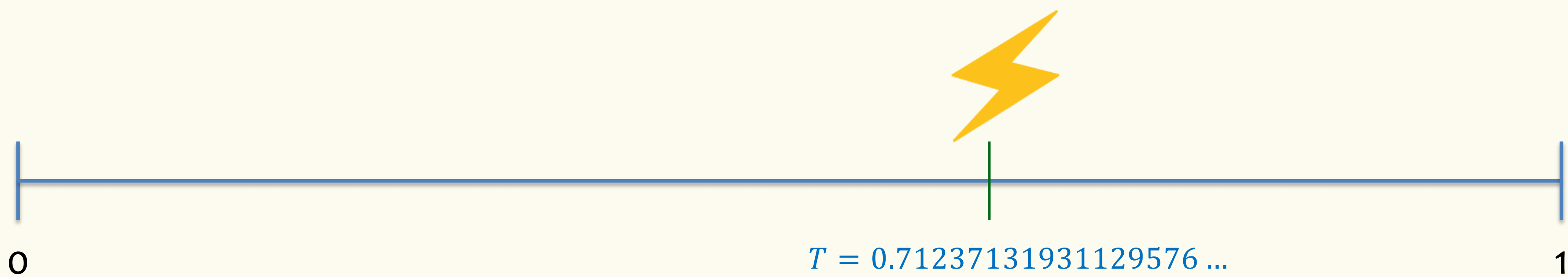
- Continuous Random Variables
- Probability Density Function
- Cumulative Density Function

Often we want to model experiments where the outcome is not discrete.

Example – Lightning Strike

Lightning strikes a pole within a one-minute time frame

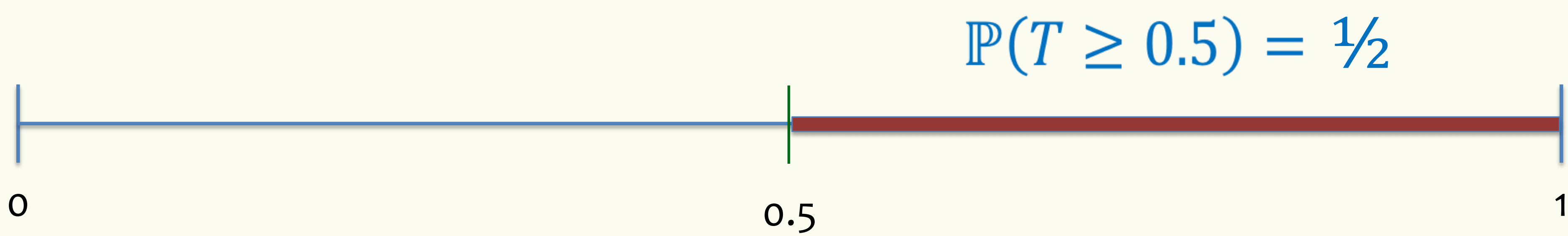
- T = time of lightning strike
- Every time within $[0,1]$ is equally likely
  - Time measured with infinitesimal precision.



The outcome space is not discrete

Lightning strikes a pole within a one-minute time frame

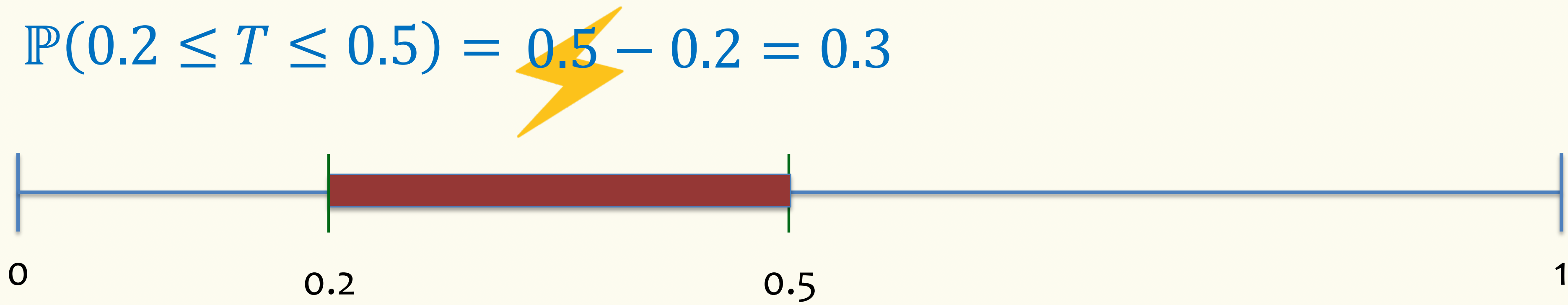
- T = time of lightning strike
- Every point in time within $[0,1]$ is equally likely



Lightning strikes a pole within a one-minute time frame

- T = time of lightning strike
- Every point in time within [0,1] is equally likely

$$\mathbb{P}(0.2 \leq T \leq 0.5) = 0.5 - 0.2 = 0.3$$



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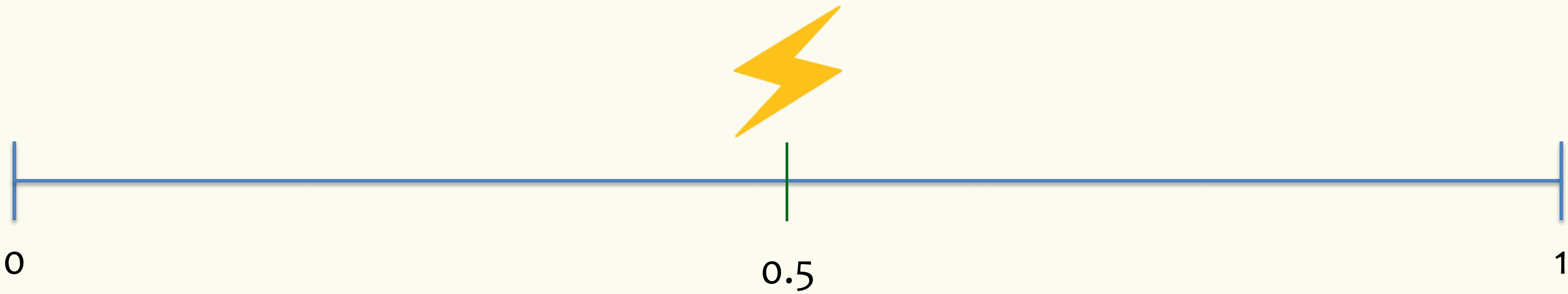

$$\mathbb{P}(T = 0.5) = 0$$

Bottom line

- This gives rise to a different type of random variable
- $\mathbb{P}(T = x) = 0$ for all $x \in [0,1]$
- Yet, somehow we want
  - $\mathbb{P}(T \in [0,1]) = 1$
  - $\mathbb{P}(T \in [a, b]) = b - a$
  - …
- How do we model the behavior of T?