CSE 312

Foundations of Computing II

Lecture 11: Variance and Independence of RVs

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Slide Credit: Based on Stefano Tessaro's slides for 312 19au

incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Recap Linearity of Expectation

$E[aX+b] = aE[X]+b$

Theorem. For any two random variables X and Y *(X, Y do not need to be independent)*

$$\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y).$$

Theorem. For any random variables $X_1, \dots, X_n$, and real numbers $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}(a_1X_1 + \cdots + a_nX_n) = a_1\mathbb{E}(X_1) + \cdots + a_n\mathbb{E}(X_n).$$

For any event A, can define the indicator random variable X for A

$$X = \begin{cases} 1 & \text{if event } A \text{ occurs} \\ 0 & \text{if event } A \text{ does not occur} \end{cases}$$

$$\mathbb{P}(X=1) = \mathbb{P}(A)$$
$$\mathbb{P}(X=0) = 1 - \mathbb{P}(A)$$

For indicator r.v. X, $E[X] = Pr(X=1)$

Recap Linearity is special!

In general $\mathbb{E}(g(X)) \neq g(\mathbb{E}(X))$

E.g., $X = \begin{cases} 1 \;\; with\ prob\ \ 1/2 \\ -1 \;\; with\ prob\ 1/2 \end{cases}$

- $\mathbb{E}(XY) \neq \mathbb{E}(X)\mathbb{E}(Y)$
- $\mathbb{E}(X/Y) \neq \mathbb{E}(X)/\mathbb{E}(Y)$
- $\boxed{\mathbb{E}(X^2) \neq \mathbb{E}(X)^2}$

How DO we compute $\mathbb{E}(g(X))$?

$E[X] = 0$

$E[X^2] = 1^2 \cdot \frac{1}{2} + (-1)^2 \cdot \frac{1}{2}$

$\therefore \; E[X^2] \neq E[X]^2$

Recap Expectation of $g(X)$

Definition. Given a discrete RV $X: \Omega \to \mathbb{R}$, the expectation or expected value of X is

$$\mathrm{E}[\overset{g(X)}{\cancel{X}}] = \sum_{\omega \in \Omega} \underline{g(X(\omega))} \cdot \Pr(\omega)$$

or equivalently

$$\mathrm{E}[\overset{g(X)}{\cancel{X}}] = \sum_{x \in X(\Omega)} g(x) \cdot \Pr(X = x)$$

Example: Expectation of $g(X)$

Remember

$E[X^2] \neq E[X]^2$

Suppose we rolled a fair, 6-sided die in a game. You will win the square number rolled dollars, times 10. Let X be the result of the dice roll. What is your expected winnings?

$$E[10X^2] = 10E[X^2]$$

$$= 10\left(1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6}\right)$$

$$= 10 \cdot \frac{91}{6} \approx 151.166$$

Agenda

- Variance
- Properties of Variance
- Independent Random Variables
- Properties of Independent Random Variables

Two Games

Game 1: *In every round, you win \$2 with probability 1/3, lose \$1 with probability 2/3.*

W_1 = payoff in a round of Game 1

$$\mathbb{P}(W_1 = 2) = \frac{1}{3}, \mathbb{P}(W_1 = -1) = \frac{2}{3}$$

$$\mathbb{E}(W_1) = 0$$

Game 2: *In every round, you win \$10 with probability 1/3, lose \$5 with probability 2/3.*

W_2 = payoff in a round of Game 2

$$\mathbb{P}(W_2 = 10) = \frac{1}{3}, \mathbb{P}(W_2 = -5) = \frac{2}{3}$$

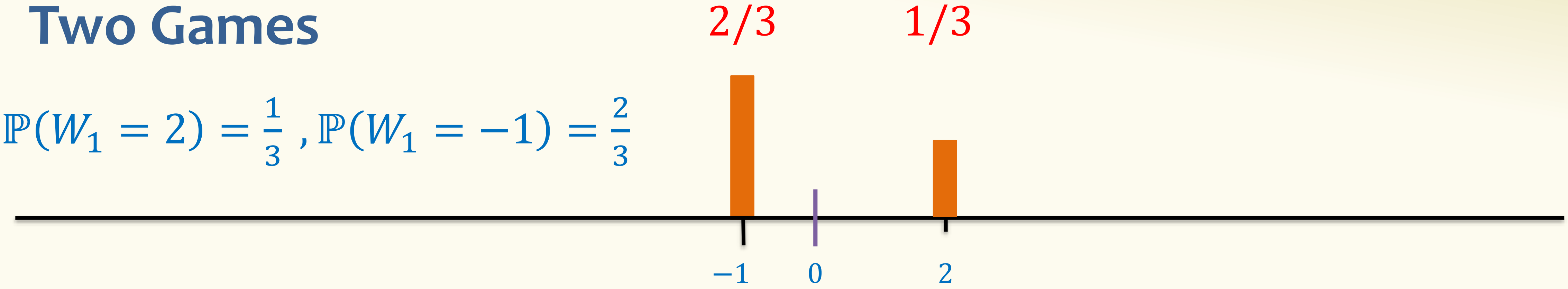
$$\mathbb{E}(W_2) = 0$$

Which game would you <u>rather</u> play?

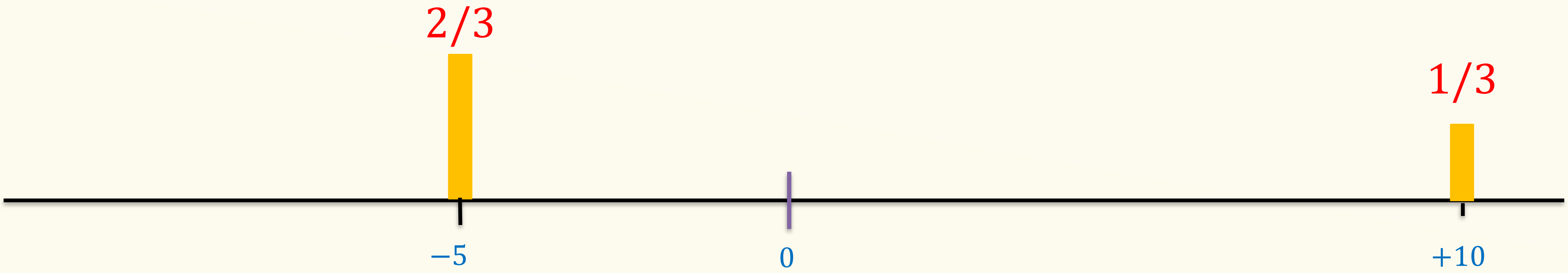
Somehow, Game 2 has higher volatility / exposure!

Two Games

$$\mathbb{P}(W_1 = 2) = \frac{1}{3}, \mathbb{P}(W_1 = -1) = \frac{2}{3}$$



$$\mathbb{P}(W_2 = 10) = \frac{1}{3}, \mathbb{P}(W_2 = -5) = \frac{2}{3}$$



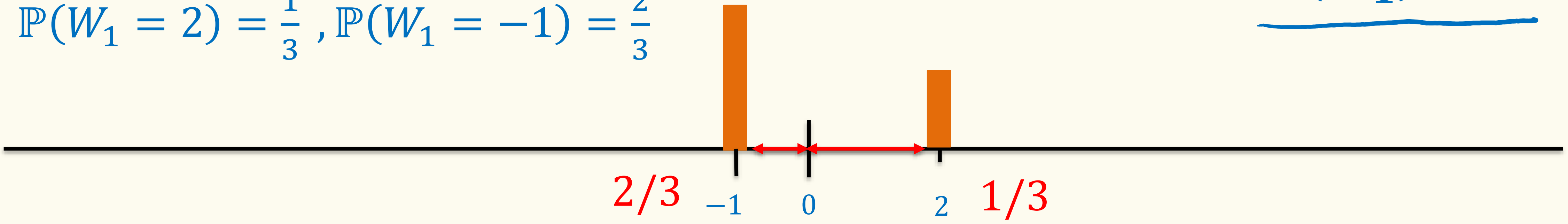
Same expectation, but clearly very different distribution.

We want to capture the difference – New concept: **Variance**

Variance (Intuition, First Try)

$$\mathbb{E}(W_1) = 0$$

$\mathbb{P}(W_1 = 2) = \frac{1}{3}$, $\mathbb{P}(W_1 = -1) = \frac{2}{3}$



New quantity (random variable): How far from the expectation?

$$\Delta(W_1) = W_1 - E[W_1]$$

$E[4] = 4$

$E[E[X]] = E[X]$

$E[X]$ is just a number

$$E[\Delta(W_1)] = E[\overbrace{W_1}^{\text{RV}} - \overbrace{E[W_1]}^{\text{Number (in this case: 0)}}]$$

$$= E[W_1] - E\big[E[W_1]\big]$$

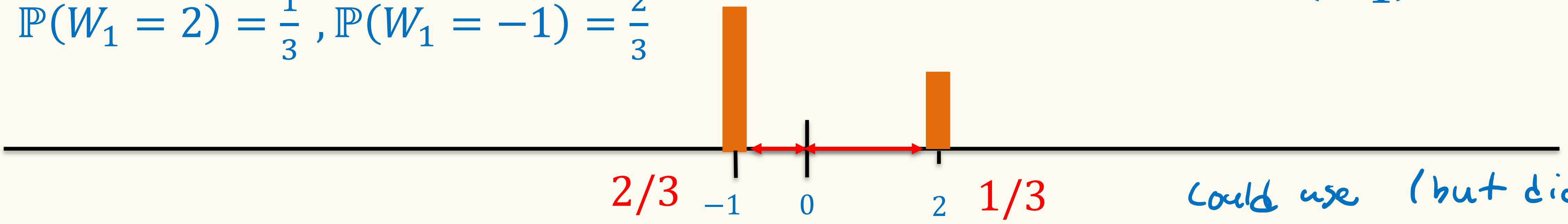
$$= E[W_1] - E[W_1]$$

$$= 0$$

Variance (Intuition, Better Try)

$$\mathbb{E}(W_1) = 0$$

$\mathbb{P}(W_1 = 2) = \frac{1}{3}$, $\mathbb{P}(W_1 = -1) = \frac{2}{3}$



Could use (but didn't) $|W_1 - E[W_1]|$

A better quantity (random variable): How far from the expectation?

$$\Delta(W_1) = (W_1 - E[W_1])^2$$

$$\mathbb{P}(\Delta(W_1) = 1) = \frac{2}{3}$$

$$\mathbb{P}(\Delta(W_1) = 4) = \frac{1}{3}$$

$$E[\Delta(W_1)] = E[(W_1 - E[W_1])^2]$$

$$= \frac{2}{3} \cdot 1 + \frac{1}{3} \cdot 4$$

$$= 2$$

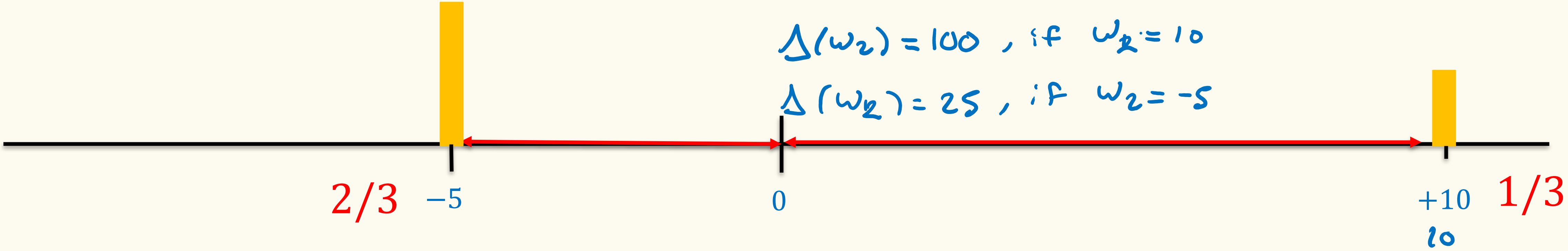
Variance (Intuition, Better Try)

$E[W_2] = 0$

$\mathbb{P}(W_2 = 10) = \frac{1}{3}, \mathbb{P}(W_2 = -5) = \frac{2}{3}$

$\Delta(w_2) = 100$, if $w_2 = 10$

$\Delta(w_2) = 25$, if $w_2 = -5$



A better quantity (random variable): How far from the expectation?

$$\Delta(W_2) = (W_2 - E[W_2])^2$$

$$\mathbb{P}(\Delta(W_2) = 25) = \frac{2}{3}$$

$$\mathbb{P}(\Delta(W_2) = 100) = \frac{1}{3}$$

$$E[\Delta(W_2)] = E[(W_2 - E[W_2])^2]$$

$$= \frac{2}{3} \cdot 25 + \frac{1}{3} \cdot 100$$

$$= 50$$

Poll:
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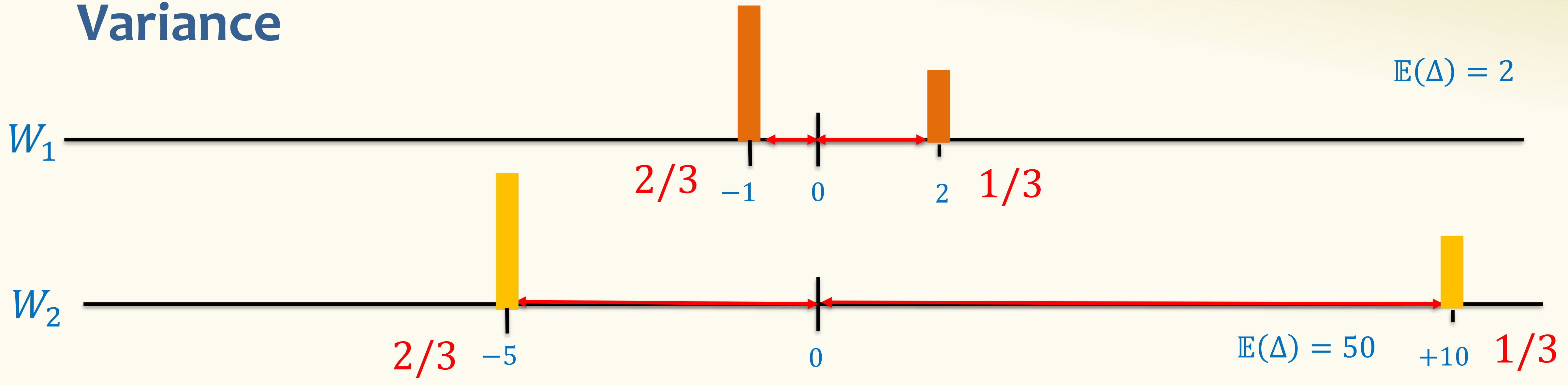
A. 0
B. 20/3
C. 50
D. 2500

Variance



We say that W_2 has **"higher variance"** than W_1.

Variance

Just a fancy $g(X)$ for $E[g(X)]$

Definition. The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right] = \sum_x \mathbb{p}_X(x) \cdot \left(x - \mathbb{E}(X)\right)^2$$

Recall $\mathbb{E}(X)$ is a **constant**, not a random variable itself.

Standard deviation: $\sigma(X) = \sqrt{\mathrm{Var}(X)}$

changes back to "units" of X

$\mathrm{Var}(X) \geq 0$ always

Intuition: Variance (or standard deviation) is a quantity that measures, in expectation, how "far" the random variable is from its expectation.

Variance – Example 1

X fair die

- $\mathbb{P}(X = 1) = \cdots = \mathbb{P}(X = 6) = 1/6$
- $\mathbb{E}(X) = 3.5$

$$\mathrm{Var}(X) = \sum_x \mathbb{P}(X = x) \cdot \overbrace{\left(x - \mathbb{E}(X)\right)^2}^{g(x)}$$

$$= \frac{1}{6}\left[(1-3.5)^2 + (2-3.5)^2 + (3-3.5)^2 + (4-3.5)^2 + (5-3.5)^2 + (6-3.5)^2\right]$$

$$= \frac{2}{6}\left[2.5^2 + 1.5^2 + 0.5^2\right] = \frac{2}{6}\left[\frac{25}{4} + \frac{9}{4} + \frac{1}{4}\right] = \frac{35}{12} \approx 2.91677\ldots$$

Variance in Pictures

Other Notation

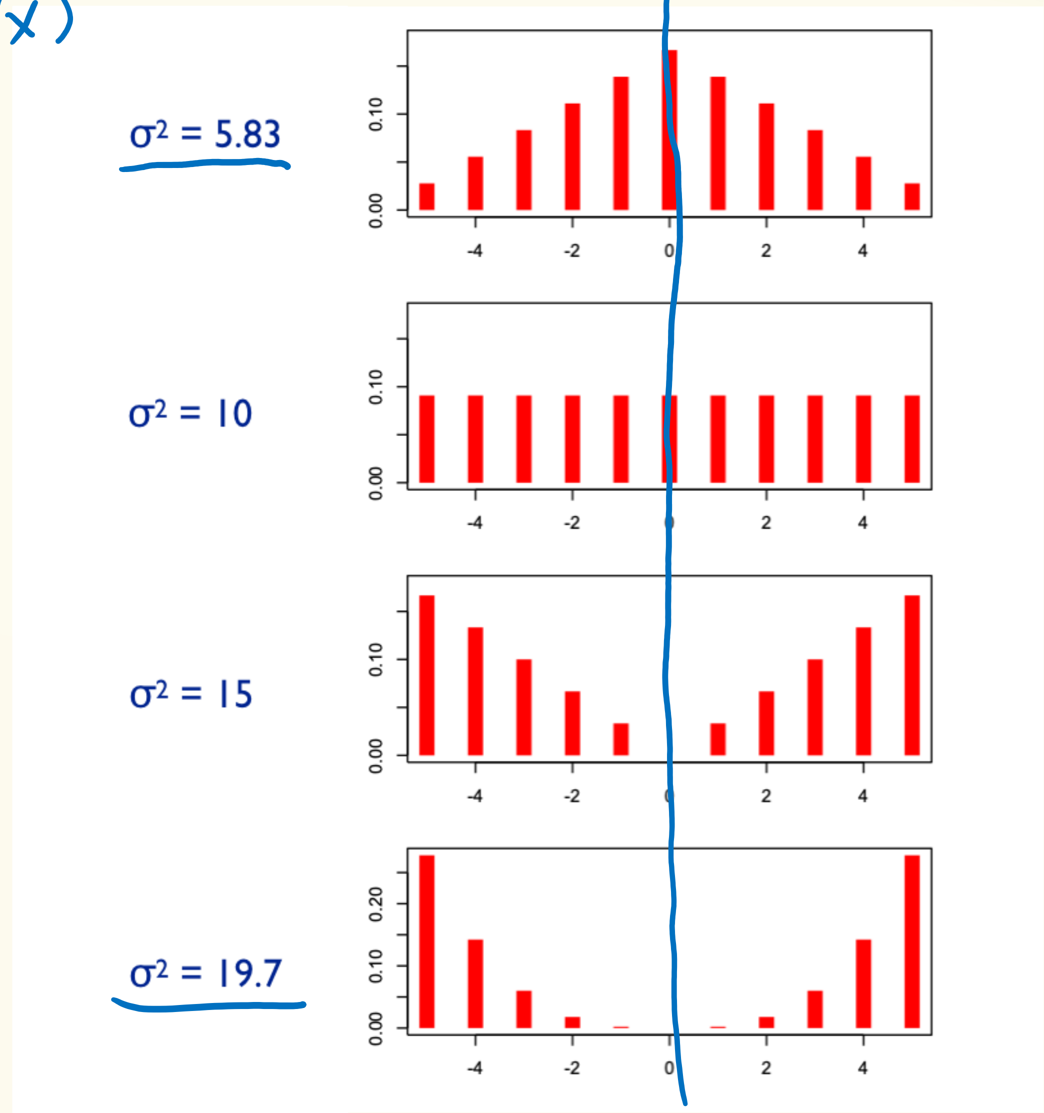
"mu" $\rightarrow$ $\mu = E[X]$

"sigma^2" $\rightarrow$ $\sigma^2 = Var(X)$

Captures how much "spread' there is in a pmf

All pmfs have same expectation

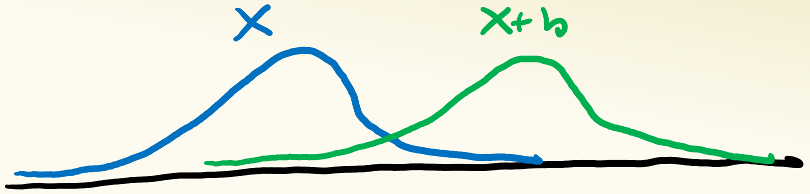
μ



Agenda

- Variance
- Properties of Variance
- Independent Random Variables
- Properties of Independent Random Variables

Variance – Properties



$\mathrm{Var}(X+b) = \mathrm{Var}(X)$

Definition. The **variance** of a (discrete) RV X is

$$\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right] = \sum_x \mathbb{p}_X(x) \cdot \left(x - \mathbb{E}(X)\right)^2$$

Theorem. For any $a, b \in \mathbb{R}$, $\mathrm{Var}(a \cdot X + b) = \underline{a^2} \cdot \mathrm{Var}(X)$

(Proof: Exercise!)

Remember: $\mathbb{E}[X^2] \neq \mathbb{E}[X]^2$

Theorem. $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

Variance

Theorem. $\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2$

Proof: $\mathrm{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}(X)\right)^2\right]$

Recall $\mathbb{E}(X)$ is a **constant**

$= \mathbb{E}[X^2 - 2\mathbb{E}(X) \cdot X + \mathbb{E}(X)^2]$

$= \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + \mathbb{E}(X)^2$

(linearity of expectation!)

$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$

$\mathbb{E}(X^2)$ and $\mathbb{E}(X)^2$ are different !

Because $E[X]$ is just a number (not RV)

$E[E[X]] = E[X]$

$E[E[X]^2] = E[X]^2$

Variance – Example 1

Usually easier to compute

$E[X]$ and $E[X^2]$

instead of

$E[(X-E[X])^2]$

X fair die

- $\mathbb{P}(X=1) = \cdots = \mathbb{P}(X=6) = 1/6$
- $\mathbb{E}(X) = \frac{21}{6}$
- $\mathbb{E}(X^2) = \frac{91}{6} = \left(1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + \ldots\right)$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{91}{6} - \left(\frac{21}{6}\right)^2 = \frac{105}{36} \approx 2.91677$$

In General, $\mathrm{Var}(X + Y) \neq \mathrm{Var}(X) + \mathrm{Var}(Y)$

Proof by counter-example:

- Let X be a r.v. with pmf $\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = 1/2$
  - What is $\mathrm{E}[X]$ and $\mathrm{Var}(X)$? $E[X] = 0$, $Var(X) = 1$
- Let $Y = -X$
  - What is $\mathrm{E}[Y]$ and $\mathrm{Var}(Y)$? $E[Y] = 0$, $Var(Y) = 1$

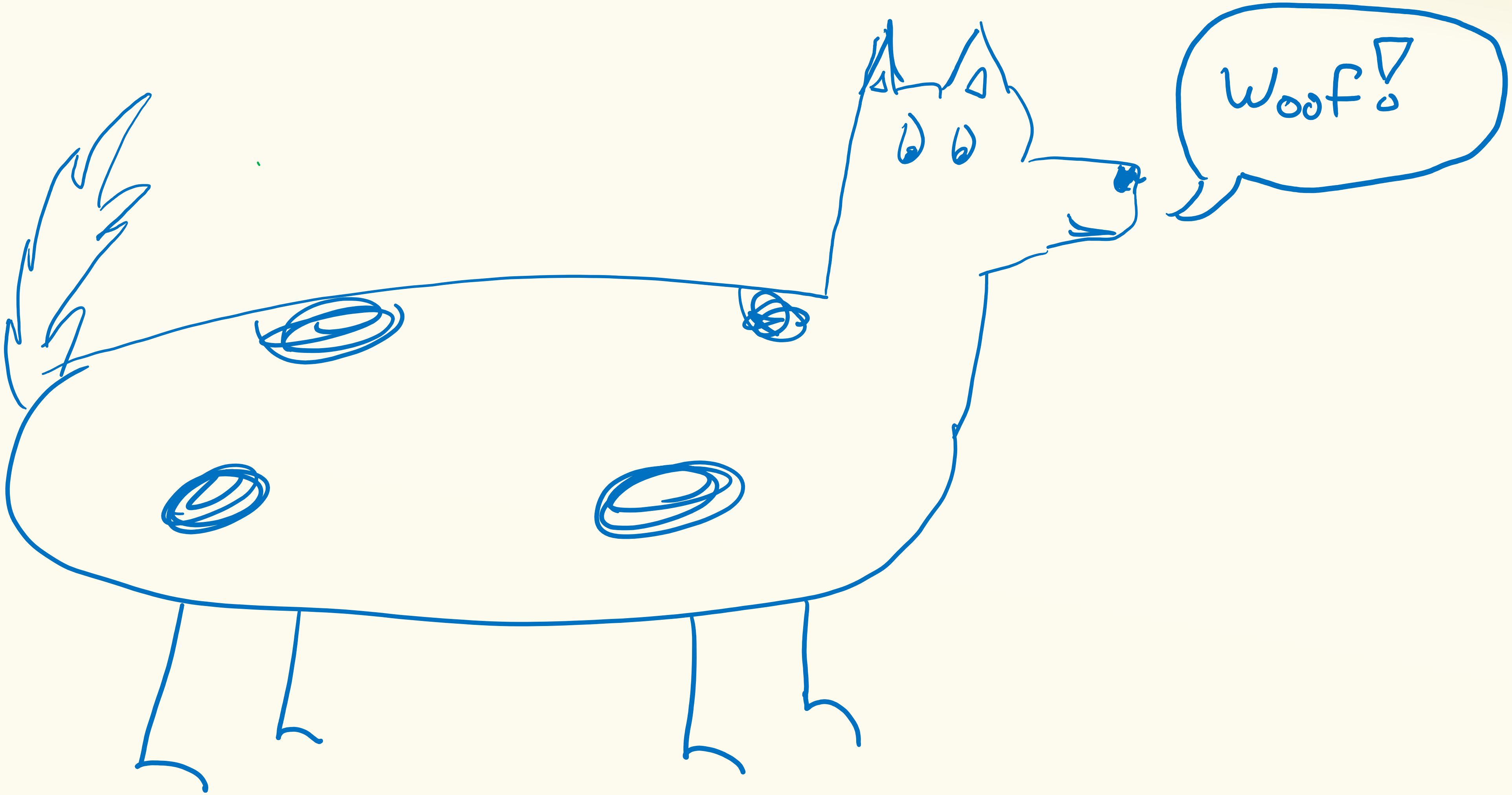
What is $\mathrm{Var}(X + Y)$?

$Var(X+Y) = Var(X-X) = Var(0) = 0$

$\therefore Var(X+Y) \neq Var(X) + Var(Y)$

$Var(X) + Var(Y) = 1 + 1 = 2$

Brain Break



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Random Variables and Independence

Recall indep. events

$P(A \cap B) = P(A)P(B)$

$Pr(X=x) = Pr(\{\omega \in \Omega \mid X(\omega) = x\})$

Definition. Two random variables X, Y are **(mutually) independent** if for all x, y,

$$\mathbb{P}(X = x, Y = y) = \mathbb{P}(X = x) \cdot \mathbb{P}(Y = y)$$

Intuition: Knowing X doesn't help you guess Y and vice versa

$Pr(X=x \mid Y=y) = Pr(X=x)$ if $P(Y=y) > 0$

Definition. The random variables $X_1, \ldots, X_n$ are **(mutually) independent** if for all $x_1, \ldots, x_n$,

$$\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) = \mathbb{P}(X_1 = x_1) \cdots \mathbb{P}(X_n = x_n)$$

Note: No need to check for all subsets, but need to check <u>for all outcomes</u>!

Example

Let X be the number of heads in n *independent* coin flips of the same coin. Let $Y = \mathrm{X} \bmod 2$ be the parity (even/odd) of X.

Are X and Y independent?

$$\Pr(X=k) = \binom{n}{k} / 2^n$$

Not independent

$\Rightarrow$ dependent

$$\Pr(X=5 \mid Y=0) \stackrel{?}{=} \Pr(X=5)$$

$$\begin{array}{ccc} \| & & \| \\ 0 & \neq & \binom{n}{5}/2^n \end{array}$$

Not possible

Poll:
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A. Yes
B. No

Example

$\underbrace{HHTHTTHHT}_{n},\ \underbrace{TTTHHHTTH}_{n}$

Make $2n$ *independent* coin flips of the same coin. Let X be the number of heads in the first n flips and Y be the number of heads in the last n flips.

Are X and Y independent?

$Pr(X=j) = \binom{n}{j}/2^n$

$Pr(Y=k) = \binom{n}{k}/2^n$

$Pr(X=x \cap Y=y) = \binom{n}{j}\binom{n}{k}/2^{2n}$

$Pr(X=x \cap Y=y) = \binom{n}{j}\binom{n}{k}/2^{2n} = \left(\binom{n}{j}/2^n\right)\left(\binom{n}{k}/2^n\right)$

$= Pr(X=x)\,Pr(Y=y)$

Poll:
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✓A. Yes
B. No

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Important Facts about Independent Random Variables

Theorem. If X, Y independent, $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$

Theorem. If X, Y independent, $\mathrm{Var}(X + Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Corollary. If $X_1, X_2, \ldots, X_n$ mutually independent,

$$\mathrm{Var}\left(\sum_{i=1}^{n} X_i\right) = \sum_{i}^{n} \mathrm{Var}(X_i)$$

(Not Covered) Proof of $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$

Theorem. If X, Y independent, $\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$

Proof

Let $x_i, y_i, i = 1, 2, \ldots$ be the possible values of X, Y.

$$
\begin{aligned}
E[X \cdot Y] &= \sum_i \sum_j x_i \cdot y_j \cdot \boxed{P(X = x_i \wedge Y = y_j)} \quad \text{independence} \\
&= \sum_i \sum_j x_i \cdot y_j \cdot \boxed{P(X = x_i) \cdot P(Y = y_j)} \\
&= \sum_i x_i \cdot P(X = x_i) \cdot \left(\sum_j y_j \cdot P(Y = y_j) \right) \\
&= E[X] \cdot E[Y]
\end{aligned}
$$

Note: *NOT* true in general; see earlier example $E[X^2] \neq E[X]^2$

(Not Covered) Proof of $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Theorem. If X, Y independent, $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$

Proof

$$Var[X+Y]$$

$$= E[(X+Y)^2] - (E[X+Y])^2$$

$$= E[X^2 + 2XY + Y^2] - (E[X] + E[Y])^2$$

$$= E[X^2] + 2E[XY] + E[Y^2] - ((E[X])^2 + 2E[X]E[Y] + (E[Y])^2)$$

$$= E[X^2] - (E[X])^2 + E[Y^2] - (E[Y])^2 + 2(\underline{E[XY]} - E[X]E[Y])$$

independence

$$= Var[X] + Var[Y] + 2(\underline{E[X]E[Y]} - E[X]E[Y])$$

$$= Var[X] + Var[Y]$$

Example – Coin Tosses

Next time!

We flip n *independent* coins, each one heads with probability p

- $X_i = \begin{cases} 1, & i\text{–th outcome is heads} \\ 0, & i\text{–th outcome is tails.} \end{cases}$
- $Z =$ number of heads

Fact. $Z = \sum_{i=1}^{n} X_i$

$\mathbb{P}(X_i = 1) = p$
$\mathbb{P}(X_i = 0) = 1 - p$

$\mathbb{P}(Z = k) = \binom{n}{k} p^k (1-p)^{n-k}$

What is $\mathrm{E}[Z]$? What is $\mathrm{Var}(Z)$?

Note: $X_1, \ldots, X_n$ are mutually independent! [Verify it formally!]

$$\mathrm{Var}(Z) = \sum_{i=1}^{n} \mathrm{Var}(X_i) = n \cdot p(1-p)$$

Note $\mathrm{Var}(X_i) = p(1-p)$