Feel free to ask Q’s in cuat before/after/during class

CSE 312
Foundations of Computing II

Lecture 9: Random Variables and Expectation

Slide Credit: Based on Stefano Tessaro’s slides for 312 19au incorporating ideas from Alex Tsun’s and Anna Karlin’s slides for 312 20su and 20au

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Music: Sylvan Eeso
Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are (statistically) independent if
$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

"Equivalently." $\mathbb{P}(\mathcal{A}|\mathcal{B}) = \mathbb{P}(\mathcal{A}).$

Definition. Two events $\mathcal{A}$ and $\mathcal{B}$ are independent conditioned on $\mathcal{C}$ if
$$\mathbb{P}(\mathcal{C}) \neq 0 \text{ and } \mathbb{P}(\mathcal{A} \cap \mathcal{B} | \mathcal{C}) = \mathbb{P}(\mathcal{A} | \mathcal{C}) \cdot \mathbb{P}(\mathcal{B} | \mathcal{C}).$$

Theorem. (Chain Rule) For events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$,
$$\mathbb{P}(\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_n) = \mathbb{P}(\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_2|\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_3|\mathcal{A}_1 \cap \mathcal{A}_2) \cdots \mathbb{P}(\mathcal{A}_n|\mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_{n-1})$$
Agenda

• Random Variables
• Probability Mass Function (PMF)
• Cumulative Distribution Function (CDF)
• Expectation
Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

- *What is the total of two dice rolls?*
- *What is the number of coin tosses needed to see the first head?*
- *What is the number of heads among 2 coin tosses?*
Random Variables

Definition. A random variable (RV) for a probability space $(\Omega, \mathbb{P})$ is a function $X: \Omega \to \mathbb{R}$. The set of values that $X$ can take on is called its range/support $X(\Omega)$.

Example. Number of heads in 2 independent coin flips $\Omega = \{HH, HT, TH, TT\}$
RV Example

\[ \Omega = \text{unordered sets of 3 balls} \]
\[ \Pr(\omega) = \frac{1}{\binom{20}{3}} = \frac{1}{1140} \]

20 balls labeled 1, 2, …, 20 in a bin

– Draw a subset of 3 uniformly at random
– Let \( X = \text{maximum of the 3 numbers on the balls} \)
  
  • Example: \( X(2, 7, 5) = 7 \)
  • Example: \( X(15, 3, 8) = 15 \)

**Poll:** pollev.com/hunter312

What is \( |\text{support } X(\Omega)| \)?

- \( \max: X(\omega) = 20 \)
- \( \min: X(\omega) = 3 \)
- All integers between possible

\[ X(\Omega) = \{3, 4, \ldots, 20\} \]

\[ X(\Omega) = \{3, 4, \ldots, 20\} \]

A. \( 20^3 \)
B. 20
C. 18
D. \( \binom{20}{3} \)
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Probability Mass Function (Idea)

Flipping two independent coins

\[ \Omega = \{HH, HT, TH, TT\} \]

\[ X(\Omega) = \{0, 1, 2\} \]

What is the support \( X(\Omega) \)?

What is the probability that \( X = 2 \)? To answer this, we introduce the notion of a probability mass function (PMF) that describes this probability.

\[
Pr(X = k) = \begin{cases} 
\frac{1}{4}, & k = 0 \\
\frac{1}{2}, & k = 1 \\
\frac{1}{4}, & k = 2 \\
0, & \text{otherwise}
\end{cases}
\]
Definition. For a RV $X : \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} \overset{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $\mathbb{P}(X = x) = \mathbb{P}(\{X = x\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$ where $\mathbb{P}(X = x)$ is the probability mass function (PMF) of $X$.

Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$
RV Example

20 balls labeled 1, 2, ..., 20 in a bin

– Draw a subset of 3 uniformly at random
– Let $X =$ maximum of the 3 numbers on the balls

$$\frac{1}{X = 20} = \frac{1}{\{\omega \in \Omega \mid X(\omega) = 20\}}$$

What is $Pr(X = 20)$?

$$Pr(X = 20) = \frac{|\{x = 20\}|}{20!} = \frac{\binom{19}{2}}{\binom{20}{3}}$$

Poll: pollev.com/hunter312

A. $\frac{\binom{20}{2}}{\binom{20}{3}}$
B. $\frac{\binom{19}{2}}{\binom{20}{3}}$ (Correct)
C. $19^2 / \binom{20}{3}$
D. $19 \cdot 18 / \binom{20}{3}$
Agenda

• Random Variables
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Cumulative Distribution Function (CDF)

**Definition.** For a RV $X: \Omega \rightarrow \mathbb{R}$, the cumulative distribution function of where $X$ specifies for any real number $x$, the probability that $X \leq x$.

$$F_X(x) = \Pr(X \leq x)$$

Go back to 2 coin clips, where $X$ is the number of heads.

$$\Pr(X = x) = \begin{cases} 
\frac{1}{4}, & x = 0 \\
\frac{1}{2}, & x = 1 \\
\frac{1}{4}, & x = 2 
\end{cases}$$

$$F_X(x) = \begin{cases} 
0, & x < 0 \\
\frac{1}{4}, & 0 \leq x < 1 \\
\frac{3}{4}, & 1 \leq x < 2 \\
1, & 2 \leq x 
\end{cases}$$

$F_X(1.5) = \Pr(X \leq 1.5) = \frac{3}{4}$
Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW.

<table>
<thead>
<tr>
<th>$Pr(\omega)$</th>
<th>$\omega$</th>
<th>$X(\omega)$</th>
</tr>
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<tbody>
<tr>
<td>$1/6$</td>
<td>1, 2, 3</td>
<td>3</td>
</tr>
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$PMF$:

$Pr(X=k) = \begin{cases} 
\frac{1}{3}, & k = 0 \\
\frac{1}{2}, & k = 1 \\
\frac{1}{6}, & k = 2 
\end{cases}$

$CDF$:

$F_x(x) = Pr(X \leq x)$
• Random Variables
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Expectation (Idea)

\[ X(\mathcal{X}) = 0, 1, 2 \]

What is the *expected* number of heads in 2 independent flips of a fair coin?

\[
E[X] = \sum \Pr(x = i) \cdot x = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1
\]

equiv.

\[
E[X] = X(\text{HH}) \Pr(\text{HH}) + X(\text{HT}) \Pr(\text{HT}) + X(\text{TH}) \Pr(\text{TH}) + X(\text{TT}) \Pr(\text{TT}) = 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = 1
\]
Definition. Given a discrete RV $X: \Omega \rightarrow \mathbb{R}$, the expectation or expected value of $X$ is

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot Pr(\omega)$$

or equivalently

$$E[X] = \sum_{x \in X(\Omega)} x \cdot Pr(X = x)$$

Intuition: “Weighted average” of the possible outcomes (weighted by probability)
Example: Returning Homeworks

Let $X$ be the number of students who get their own HW

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$$E[X] = \sum_{\omega \in \Omega} X(\omega) \cdot Pr(\omega)$$

$$= 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} = 1$$

Both ways compute same value!
Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability $p$ of being heads. Keep flipping independent flips until heads. Let $X$ be the number of flips until heads.

$$\Omega = \frac{1}{2} H, TH, TTH, \ldots$$

$x(\omega)$: 1, 2, 3, ... $x(\Omega) = \mathbb{E}[1, \infty)$

What is: $\Pr(X = 1) = p$

What is: $\Pr(X = 2) = (1-p)p$

What is: $\Pr(X = k) = (1-p)^{k-1}p$
Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability $p$ of being heads. Keep flipping independent flips until heads. Let $X$ be the number of flips until heads. What is $E[X]$?

$$E[X] = \sum_{k=1}^{\infty} k \Pr(X=k) = \sum_{k=1}^{\infty} k(1-p)^{k-1} p = \ldots = \frac{1}{p}$$

Didn't prove this, proof is extra if curious

**Extra:**
1. Use geometric series, for $0 < x < 1$, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$
2. Take $\frac{1}{x}$ of both sides, $\sum_{k=0}^{\infty} kx^k = \frac{1}{(1-x)^2}$. Looks a lot like formula above!
3. $E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p = p \cdot \frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$