Feel free to ask Q's in chat before/after/during class

# CSE 312

# Foundations of Computing II

## Lecture 9: Random Variables and Expectation

Prob Space $(\Omega, \Pr(\cdot))$ Unif Prob Space $(\Omega, \frac{1}{|\Omega|})$

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Music: Sylvan Esso

# Last Time

**Theorem. (Chain Rule)** For events $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$ ,

$$\mathbb{P}(\mathcal{A}_1 \cap \cdots \cap \mathcal{A}_n) = \mathbb{P}(\mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_2 | \mathcal{A}_1) \cdot \mathbb{P}(\mathcal{A}_3 | \mathcal{A}_1 \cap \mathcal{A}_2)$$
$$\cdots \mathbb{P}(\mathcal{A}_n | \mathcal{A}_1 \cap \mathcal{A}_2 \cap \cdots \cap \mathcal{A}_{n-1})$$

**Definition.** Two events $\mathcal{A}$ and $\mathcal{B}$ are (statistically) **independent** if

$$\mathbb{P}(\mathcal{A} \cap \mathcal{B}) = \mathbb{P}(\mathcal{A}) \cdot \mathbb{P}(\mathcal{B}).$$

**"Equivalently."** $\mathbb{P}(\mathcal{A} | \mathcal{B}) = \mathbb{P}(\mathcal{A}).$

**Definition.** Two events $\mathcal{A}$ and $\mathcal{B}$ are **independent** conditioned on $\mathcal{C}$ if
$\mathbb{P}(\mathcal{C}) \neq 0$ and $\mathbb{P}(\mathcal{A} \cap \mathcal{B} \mid \mathcal{C}) = \mathbb{P}(\mathcal{A} \mid \mathcal{C}) \cdot \mathbb{P}(\mathcal{B} \mid \mathcal{C}).$

# Agenda

- Random Variables
- Probability Mass Function (PMF)
- Cumulative Distribution Function (CDF)
- Expectation

# Random Variables (Idea)

Often: We want to **capture quantitative properties** of the outcome of a random experiment, e.g.:

– *What is the total of two dice rolls?*

– *What is the number of coin tosses needed to see the first head?*

– *What is the number of heads among 2 coin tosses?*

# Random Variables

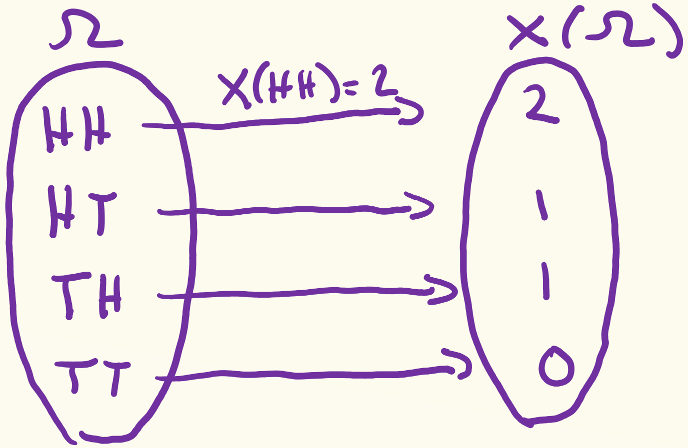
Discrete RV: $X(\Omega)$ is finite or countably infinite (e.g. integers)

**Definition.** A **random variable (RV)** for a probability space $(\Omega, \mathbb{P})$ is a function $\underline{X:}\ \Omega \to \mathbb{R}$.

Notation in Book: $\Omega_X = X(\Omega)$

The set of values that $X$ can take on is called its range/support $X(\Omega)$

**Example.** Number of heads in 2 independent coin flips $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$




Support

$X(\Omega) = \{0, 1, 2\}$

# RV Example

$\Omega$ = unordered sets of 3 balls

$Pr(\omega) = \frac{1}{|\Omega|} = \frac{1}{\binom{20}{3}}$

20 balls labeled 1, 2, …, 20 in a bin

- Draw a subset of 3 uniformly at random
- Let $X$ = maximum of the 3 numbers on the balls
  - Example: $X(2, 7, 5) = 7$
  - Example: $X(15, 3, 8) = 15$

$X(1, 2, 3) = 3$

Poll: pollev.com/hunter312

What is |support| = $|X(\Omega)|$?

- max: $X(\omega) = 20$
- min: $X(\omega) = 3$
- All integers between possible

$X(\Omega) = \{3, 4, \dots, 20\}$

A. $20^3$

B. $20$

✓C. $18$

D. $\binom{20}{3}$

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# Probability Mass Function (Idea)

Flipping two independent coins $\Omega = \{\mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT}\}$

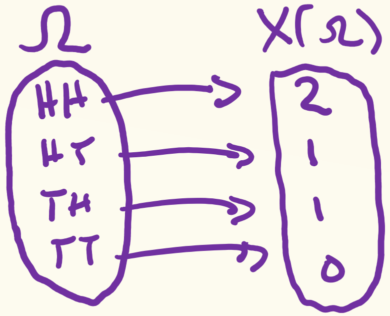
$X$ = number of heads in the two flips

$X(HH) = 2 \qquad X(HT) = 1 \qquad X(TH) = 1 \qquad X(TT) = 0$

What is the support $X(\Omega)$? $X(\Omega) = \{0, 1, 2\}$

Intuition: $1/4$

What is the probability that $X$ is 2? To answer this, we introduce the notion of a **probability mass function (PMF)** that describes this probability.




$Pr(X = k)$

$$Pr(X=k) = \begin{cases} 1/4, & k=0 \\ 1/2, & k=1 \\ 1/4, & k=2 \\ 0, & \text{otherwise} \end{cases}$$

# Probability Mass Function (PMF)

Notation in Book

$p_X(k) = \Pr(X = k)$

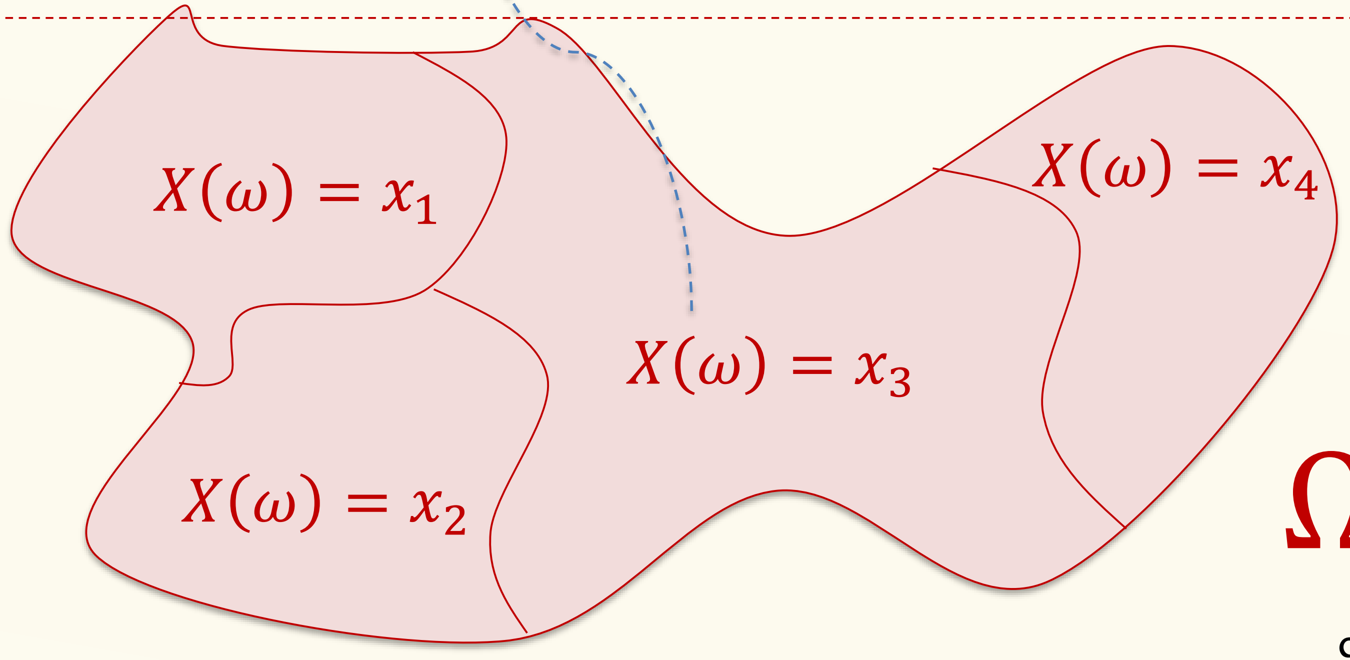
**Definition.** For a RV $X: \Omega \to \mathbb{R}$, we define the event

$$\{X = x\} \stackrel{\text{def}}{=} \{\omega \in \Omega \mid X(\omega) = x\}$$

We write $\mathbb{P}(X = x) = \mathbb{P}(\{X = x\}) = \mathbb{P}(\{\omega \in \Omega \mid X(\omega) = x\})$ where $\mathbb{P}(X = x)$ is the probability mass function (PMF) of $X$

Random variables partition the sample space.

$$\sum_{x \in X(\Omega)} \mathbb{P}(X = x) = 1$$




$\Omega$

# RV Example

General

$$Pr(X=k) = \begin{cases} \binom{k-1}{2} / \binom{20}{3} &, k = 3, 4, \ldots, 20 \\ 0 &, \text{otherwise} \end{cases}$$

20 balls labeled 1, 2, …, 20 in a bin

- Draw a subset of 3 uniformly at random
- Let $X$ = maximum of the 3 numbers on the balls

$$\{X=20\} = \{\omega \in \Omega \mid X(\omega) = 20\}$$

What is $Pr(X = 20)$?

$$Pr(X=20) = \frac{|\{X=20\}|}{|\Omega|} = \frac{\binom{19}{2}}{\binom{20}{3}}$$

Poll: pollev.com/hunter312

A. $\binom{20}{2} / \binom{20}{3}$

✓ B. $\binom{19}{2} / \binom{20}{3}$

C. $19^2 / \binom{20}{3}$

D. $19 \cdot 18 / \binom{20}{3}$


gifak-net
$h_2$
$h_1$
$b$
$a$
$S=a*h_1$
$S=b*h_2$

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# Cumulative Distribution Function (CDF)

**Definition.** For a RV $X: \Omega \to \mathbb{R}$, the cumulative distribution function of where $X$ specifies for any real number $x$, the probability that $X \leq x$.

$$\mathrm{F}_X(x) = \Pr(X \leq x)$$

$F_x(1.5) = \Pr(X \leq 1.5) = 3/4$

Go back to 2 coin clips, where $X$ is the number of heads

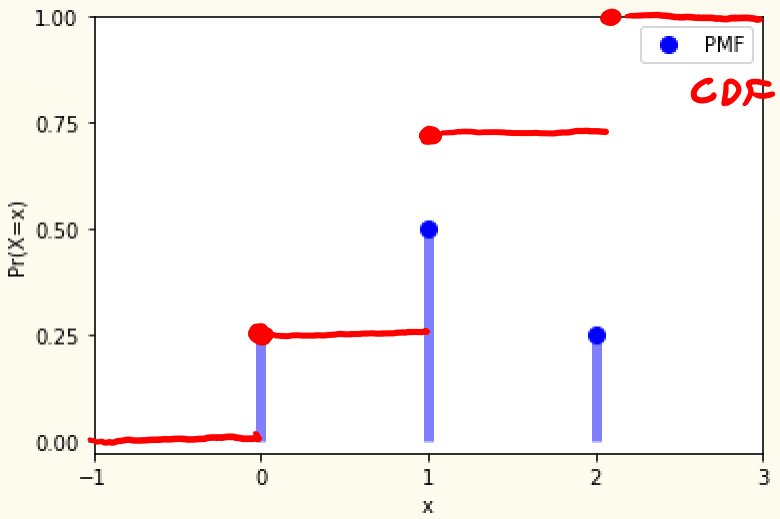
PMF

$$\Pr(X = x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \\ \frac{1}{4}, & x = 2 \end{cases}$$

CDF

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{4}, & 0 \leq x < 1 \\ \frac{3}{4}, & 1 \leq x < 2 \\ 1, & 2 \leq x \end{cases}$$

$\Pr(X \leq x)$



# Example: Returning Homeworks

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

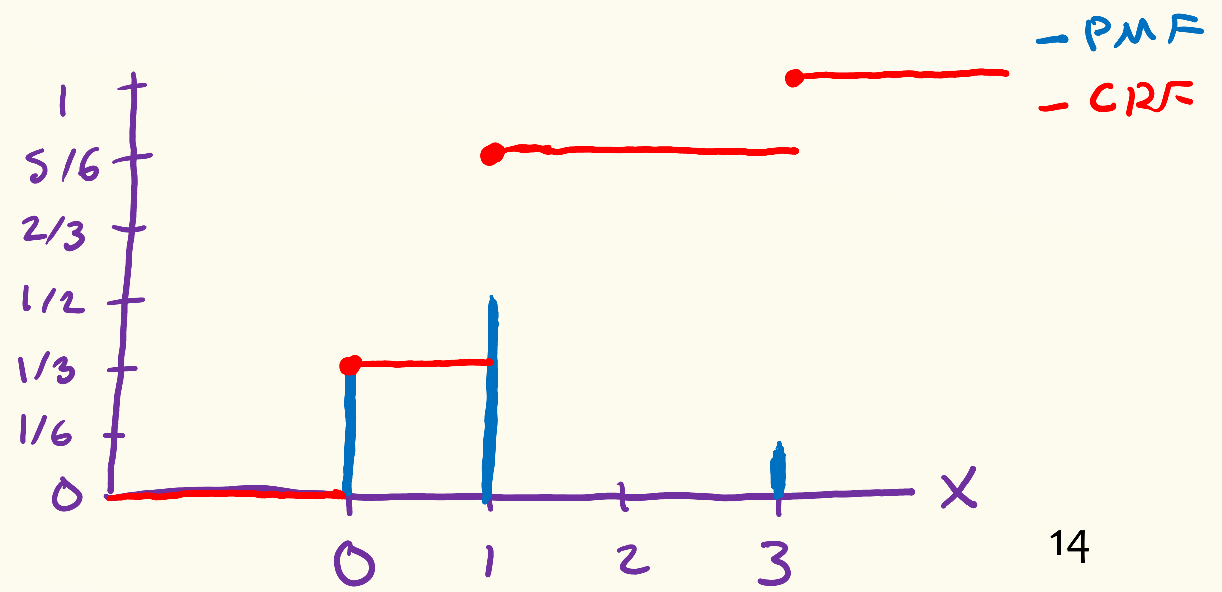
PMF

$$\Pr(X=k) = \begin{cases} 1/3, & k=0 \\ 1/2, & k=1 \\ 1/6, & k=2 \end{cases}$$

| $\Pr(\omega)$ | $\omega$ | $X(\omega)$ |
|---|---|---|
| 1/6 | 1, 2, 3 | 3 |
| 1/6 | 1, 3, 2 | 1 |
| 1/6 | 2, 1, 3 | 1 |
| 1/6 | 2, 3, 1 | 0 |
| 1/6 | 3, 1, 2 | 0 |
| 1/6 | 3, 2, 1 | 1 |

CDF

$F_X(x) = \Pr(X \le x)$



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# Expectation (Idea)

$X(\Omega) = \{0, 1, 2\}$

What is the *expected* number of heads in 2 independent flips of a fair coin?

$$E[X] = 0 \cdot \Pr(X=0) + 1 \cdot \Pr(X=1) + 2 \cdot \Pr(X=2)$$
$$= 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

equiv.

$$E[X] = X(HH)\Pr(HH) + X(HT)\Pr(HT) + X(TH)\Pr(TH) + X(TT)\Pr(TT)$$
$$= 2 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} = 1$$

# Cumulative Disribution Function (CDF)

**Definition.** Given a discrete RV $X: \Omega \to \mathbb{R}$, the expectation or expected value of $X$ is

$$\mathrm{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr(\omega)$$

or equivalently

$$\mathrm{E}[X] = \sum_{x \in X(\Omega)} x \cdot \Pr(X = x)$$

Intuition: “Weighted average” of the possible outcomes (weighted by probability)

# Example: Returning Homeworks

$X(\Omega) = \{0, 1, 3\}$

- Class with 3 students, randomly hand back homeworks. All permutations equally likely.
- Let $X$ be the number of students who get their own HW

| $\Pr(\omega)$ | $\omega$ | $X(\omega)$ |
|---|---|---|
| 1/6 | 1, 2, 3 | 3 |
| 1/6 | 1, 3, 2 | 1 |
| 1/6 | 2, 1, 3 | 1 |
| 1/6 | 2, 3, 1 | 0 |
| 1/6 | 3, 1, 2 | 0 |
| 1/6 | 3, 2, 1 | 1 |

$$E[X] = 0 \cdot \Pr(X=0) + 1 \cdot \Pr(X=1) + 3 \cdot \Pr(X=3)$$
$$= 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{2} + 3 \cdot \frac{1}{6} = 1$$

$$E[X] = \sum_{\omega \in \Omega} X(\omega) \Pr(\omega)$$
$$= 3 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 0 \cdot \frac{1}{6} + 1 \cdot \frac{1}{6} = 1$$

Both ways compute same value!

# Flip a Biased Coin Until Heads (Independent Flips)

Suppose a coin has probability $p$ of being heads. Keep flipping independent flips until heads. Let $X$ be the number of flips until heads.

$\Omega = \{H, TH, TTH, \ldots\}$

$X(\omega) \quad 1 \quad 2 \quad 3 \quad 4 \ldots$

$X(\Omega) = [1, \infty)$

What is: $\Pr(X = 1) = p$

$\{X = k\} = \{\underbrace{TTT\ldots T}_{k-1}\underbrace{H}_{1}\}$

What is: $\Pr(X = 2) = (1-p)p$

$P(TTT\ldots TH) = (1-p)^{k-1}p$

What is: $\Pr(X = k) = (1-p)^{k-1}p$

# Flip a Biased Coin Until Heads (Independent Flips)

$p = \frac{1}{20}$

Suppose a coin has probability $p$ of being heads. Keep flipping independent flips until heads. Let $X$ be the number of flips until heads. What is $\mathrm{E}[X]$?

Didn't prove this; proof is extra if curious

$$E[X] = \sum_{k=1}^{\infty} k \Pr(X=k) = \sum_{k=1}^{\infty} k(1-p)^{k-1} p = \ldots = \frac{1}{p}$$

Extra:

- Use geometric series, for $0 < x < 1$, $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$
- Take $d/dx$ of both sides, $\sum_{k=1}^{\infty} k x^k = \frac{1}{(1-x)^2}$. Looks a lot like formula above!
- $E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} p = p \cdot \frac{1}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$ ∎

Use $x = 1-p$