CSE 312

Foundations of Computing II

Lecture 6: Conditional Probability and Bayes Theorem

PAUL G. ALLEN SCHOOL OF COMPUTER SCIENCE & ENGINEERING

Rachel Lin, Hunter Schafer

Slide Credit: Based on Stefano Tessaro's slides for 312 19au
incorporating ideas from Alex Tsun's and Anna Karlin's slides for 312 20su and 20au

Announcement

- PSet 1 due tonight
  - Submit both coding and written portion on Gradescope.
  - If working in a pair, remember to add your partner to your submissions!
- PSet 2 posted on website, due next Thursday
- No class or OH on Monday 1/18 (MLK Day)

Review Probability

Definition. A **sample space** Ω is the set of all possible outcomes of an experiment.

Examples:

- Single coin flip: $\Omega = \{H, T\}$
- Two coin flips: $\Omega = \{HH, HT, TH, TT\}$
- Roll of a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

Definition. An **event** $E \subseteq \Omega$ is a subset of possible outcomes.

Examples:

- Getting at least one head in two coin flips: $E = \{HH, HT, TH\}$
- Rolling an even number on a die : $E = \{2, 4, 6\}$

Probability Space: $(\Omega, Pr(\cdot))$

$Pr(\omega) \to [0,1]$

Unif Prob space: $(\Omega, Pr(\cdot) = \frac{1}{|\Omega|})$

$Pr(\omega) = \frac{1}{|\Omega|}$, $\omega \in \Omega$

Def. (generally true; if unif. prob space)

$$Pr(E) = \sum_{\omega \in E} Pr(\omega) = \frac{|E|}{|\Omega|}$$

Review Axioms of Probability

Let Ω denote the sample space and $E, F \subseteq \Omega$ be events. Note this is more general to **any** probability space (not just uniform)

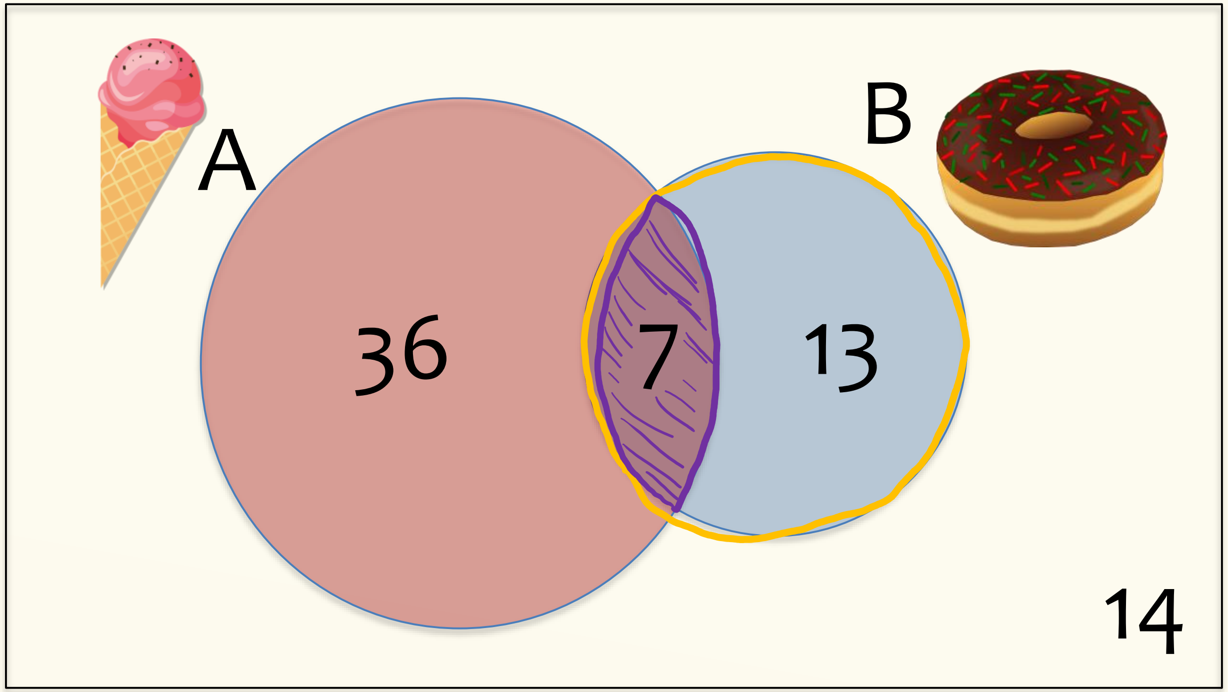
Axiom 1 (Non-negativity): $P(E) \geq 0$
Axiom 2 (Normalization): $P(\Omega) = 1$
Axiom 3 (Countable Additivity): If E and F are mutually exclusive,
then $P(E \cup F) = P(E) + P(F)$

Corollary 1 (Complementation): $P(E^c) = 1 - P(E)$
Corollary 2 (Monotonicity): If $E \subseteq F$, $P(E) \leq P(F)$
Corollary 3 (Inclusion-Exclusion): $P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Conditional Probability (Idea)



Unif Prob Space:

$$Pr(\omega) = \frac{1}{|\Omega|} = \frac{1}{70}$$

What's the probability that someone likes ice cream (A) **given** they like donuts (B)?

$$Pr(A|B) = \frac{7}{7+13} = \frac{|A \cap B|}{|B|} = \frac{|A \cap B| / |\Omega|}{|B| / |\Omega|} = \frac{Pr(A \cap B)}{Pr(B)}$$

Conditional Probability

Definition. The **conditional probability** of event *A* **given** an event *B* happened (assuming $P(B) \neq 0$) is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Read: "Prob of A given B"

An equivalent and useful formula is

Use both formulas a lot!
· Will probably memorize

$$P(A \cap B) = P(A|B)P(B)$$

Conditional Probability Examples

In the popular, social video game Among Us, you are either a crewmate or an imposter. This game, you are an imposter. What is the probability you will win the game given that you are imposter?

W = You win a game

I = You are the imposter in a game

$$P(W|I) = \frac{P(W \cap I)}{P(I)}$$

Reversing Conditional Probability

Question: Does $P(A|B) = P(B|A)$?

No!

- Let A be the event you are wet
- Let B be the event you are swimming

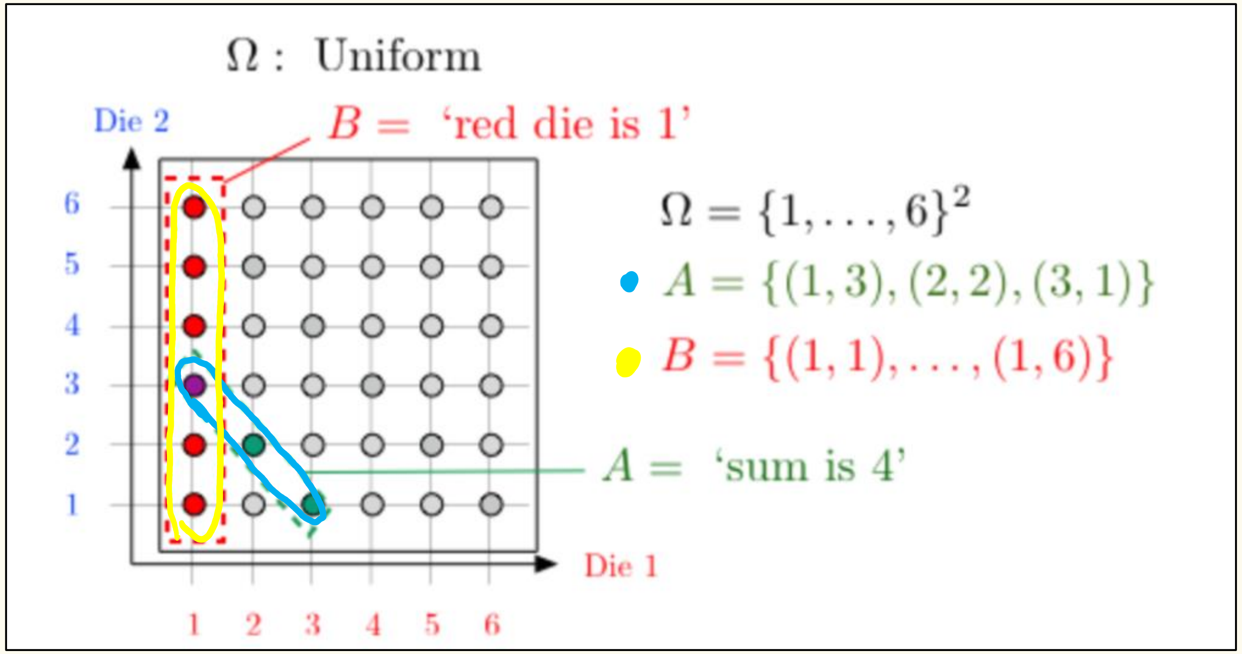
$$P(A|B) = 1$$
$$P(B|A) \neq 1$$

Example with Conditional Probability

pollev.com/hunter312

Toss a red die and a blue die (both 6 sided and all outcomes equally likely). What is $P(B)$? What is $P(B|A)$?

	P(b)	$P(B\|A)$
a)	1/6	1/6
b)	1/6	1/3
c)	1/6	3/36
d)	1/9	1/3



$$Pr(B) = \frac{|B|}{|\Omega|} = \frac{6}{36} = \frac{1}{6}$$

$$Pr(B|A) = \frac{Pr(A \cap B)}{Pr(A)} = \frac{1/36}{3/36} = \frac{1}{3}$$

Gambler's fallacy

$\Omega = \{H, T\}^{51}$ All seq. of H/T of length 50

Assume we toss **51** fair coins.
Assume we have seen **50** coins, and they are all "tails".
What are the odds the **51**st coin is "heads"?

$Pr(\omega) = \frac{1}{2^{51}}$

$\mathcal{A}$ = first 50 coins are "tails" $= \{TTT...T, TTT....H\}$

B = first 50 coins are "tails", 51st coin is "heads" $= \{TTT...H\}$

51st coin is independent of outcomes of first 50 tosses!

$$\mathbb{P}(\mathcal{B}|\mathcal{A}) = \frac{\mathbb{P}(\mathcal{A} \cap \mathcal{B})}{\mathbb{P}(\mathcal{A})} = \frac{1/2^{51}}{2/2^{51}} = \frac{1}{2}$$

Gambler's fallacy = Feels like it's time for " heads"!?

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Bayes Theorem

"Top 10" formulas in probs/stats
• Foundation for approaches in ML

A formula to let us "reverse" the conditional.

Theorem. (Bayes Rule) For events A and B, where $P(A), P(B) > 0$,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- $P(A)$ is called the **prior** (our belief without knowing anything)
- $P(A|B)$ is called the **posterior** (our belief after learning B)

Bayes Theorem Proof

$$Pr(B|A)P(A) = Pr(B \cap A) = Pr(A \cap B) = Pr(A|B)P(B)$$

$$\frac{Pr(B|A)Pr(A)}{Pr(B)} = \frac{Pr(A|B)Pr(B)}{Pr(B)}$$

$$\boxed{Pr(A|B) = \frac{Pr(B|A)\,Pr(A)}{Pr(B)}}$$

Bayes Theorem Proof

By definition of conditional probability

$$P(A \cap B) = P(A|B)P(B)$$

Swapping A, B gives

$$P(B \cap A) = P(B|A)P(A)$$

But $P(A \cap B) = P(B \cap A)$, so

$$P(A|B)P(B) = P(B|A)P(A)$$

Dividing both sides by $P(B)$ gives

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

Our First Machine Learning Task: Spam Filtering

S = is spam   F = has "FREE"

Subject: "FREE $$$ CLICK HERE"

Goal: $P(S \mid F)$

What is the probability this email is spam, given the subject contains "FREE"?

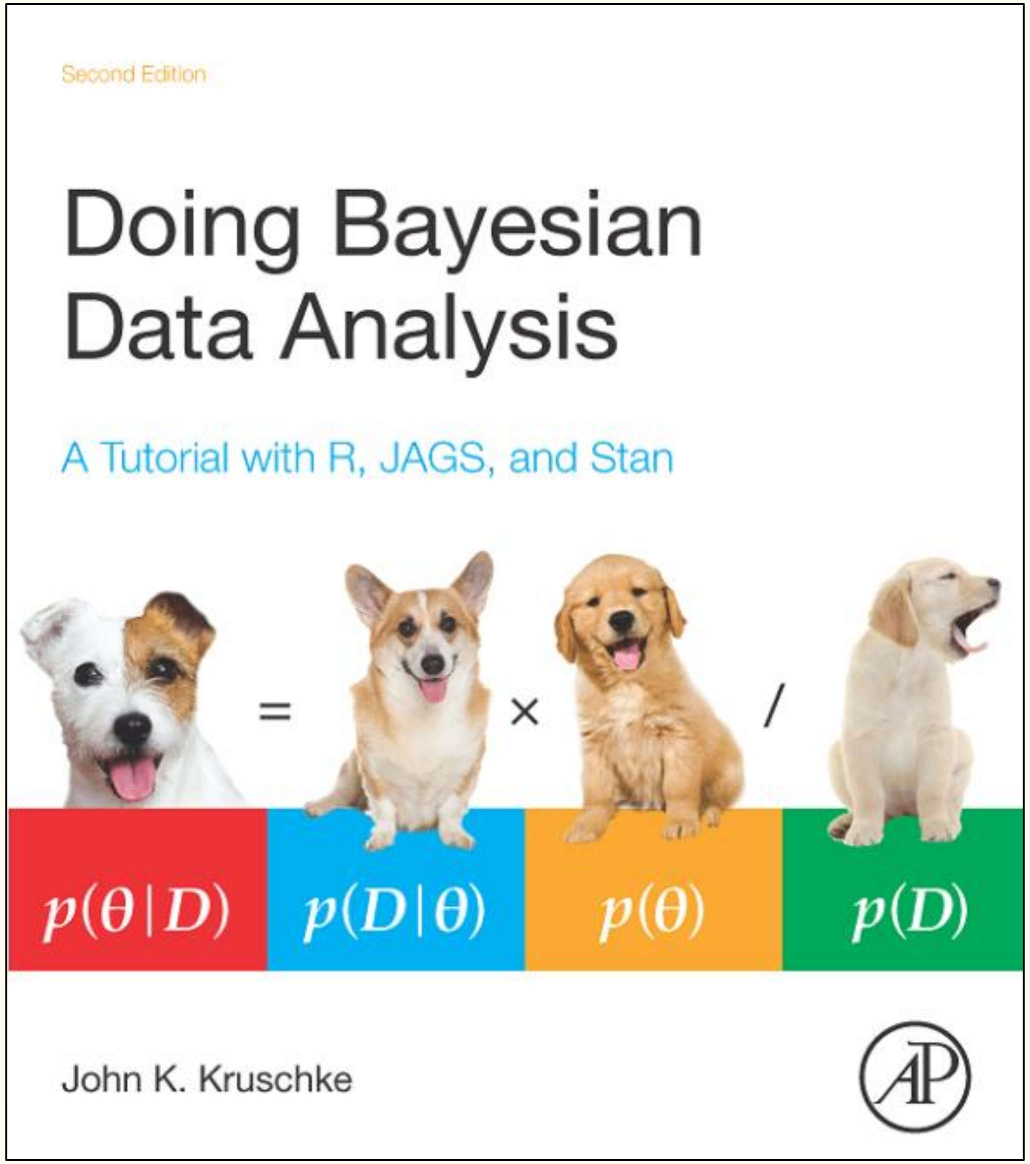
Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word "FREE" in the subject. $Pr(F \mid \bar{S}) = 0.1$
- 70% of spam emails contain the word "FREE" in the subject. $Pr(F \mid S) = 0.7$
- 80% of emails you receive are spam. $Pr(S) = 0.8$

$$Pr(S \mid F) = \frac{P(F \mid S)\, P(S)}{P(F)} = \frac{0.7 \times 0.8}{???}$$

Problem: Don't know how to compute $P(F)$!!!

Brain Break


Second Edition
Doing Bayesian Data Analysis
A Tutorial with R, JAGS, and Stan
=
×
/
p(θ|D)
p(D|θ)
p(θ)
p(D)
John K. Kruschke

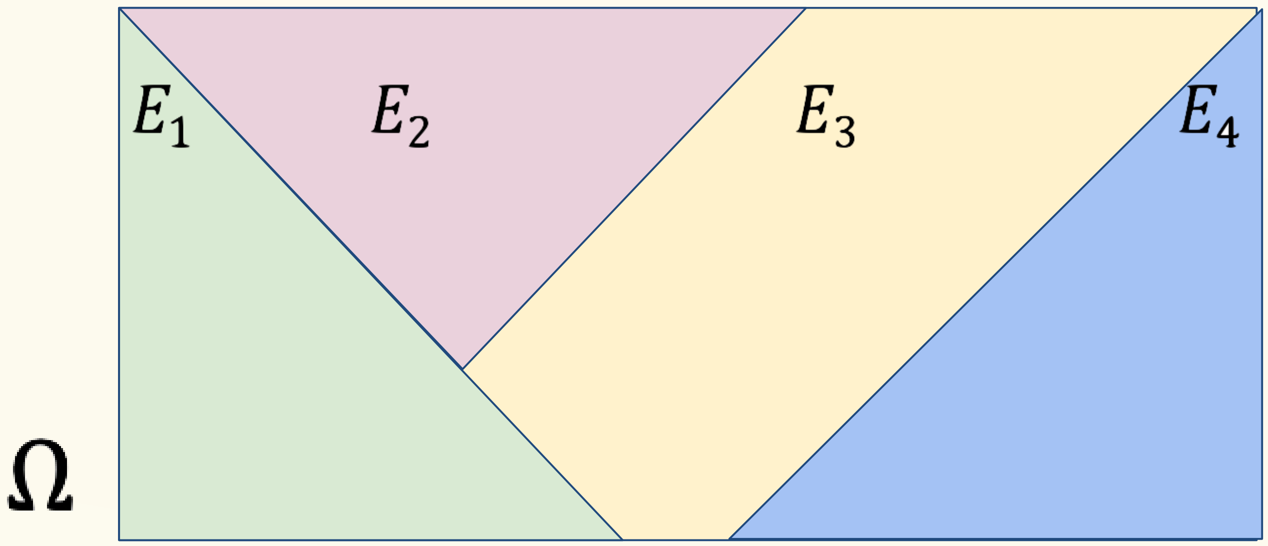
Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Partitions (Idea)

These events **partition** the sample space

1. They "cover" the whole space
2. They don't overlap



Partition

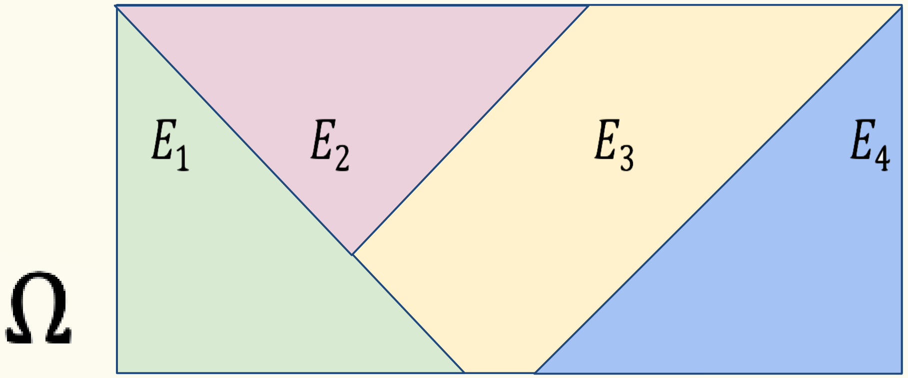
Definition. Non-empty events $E_1, E_2, \dots, E_n$ **partition** the sample space Ω if

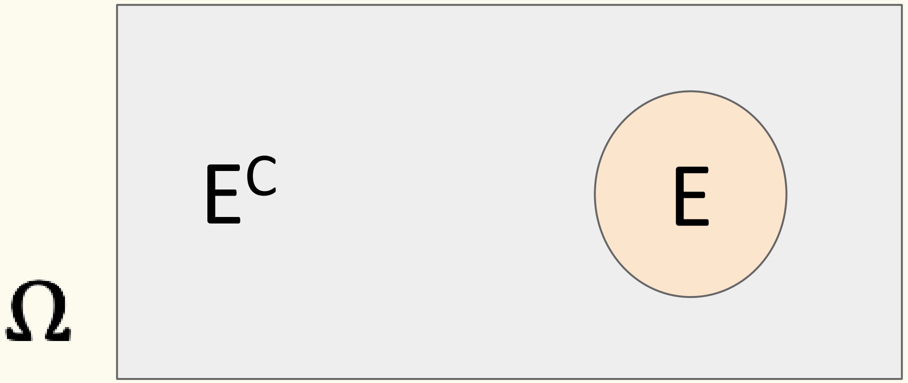
(Exhaustive)

$$E_1 \cup E_2 \cup \cdots \cup E_n = \bigcup_{i=1}^{n} E_i = \Omega$$

(Pairwise Mutually Exclusive)

$$\forall_i \forall_{i \neq j} E_i \cap E_j = \emptyset$$

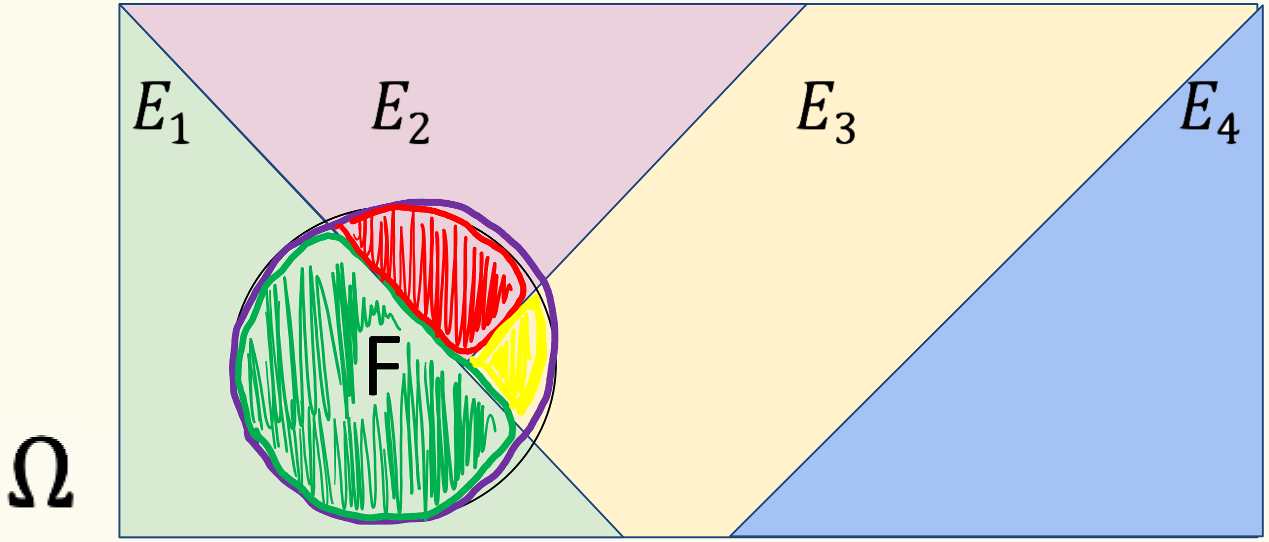




Law of Total Probability (Idea)

If we know $E_1, E_2, \dots, E_n$ partition Ω, what can we say about $P(F)$

$P(F) = P(F \cap E_1) + P(F \cap E_2) + P(F \cap E_3) + P(F \cap E_4)$ → 0



Law of Total Probability (LTP)

Definition. If events $E_1, E_2, \dots, E_n$ partition the sample space Ω, then for any event F

$$P(F) = P(F \cap E_1) + \dots + P(F \cap E_n) = \sum_{i=1}^{n} P(F \cap E_i)$$

Using the definition of conditional probability $P(F \cap E) = P(F|E)P(E)$

We can get the alternate form of this that show

$$P(F) = \sum_{i=1}^{n} P(F|E_i)P(E_i)$$

Another Contrived Example

Alice has two pockets:

- **Left pocket:** Two red balls, two green balls
- **Right pocket:** One red ball, two green balls.

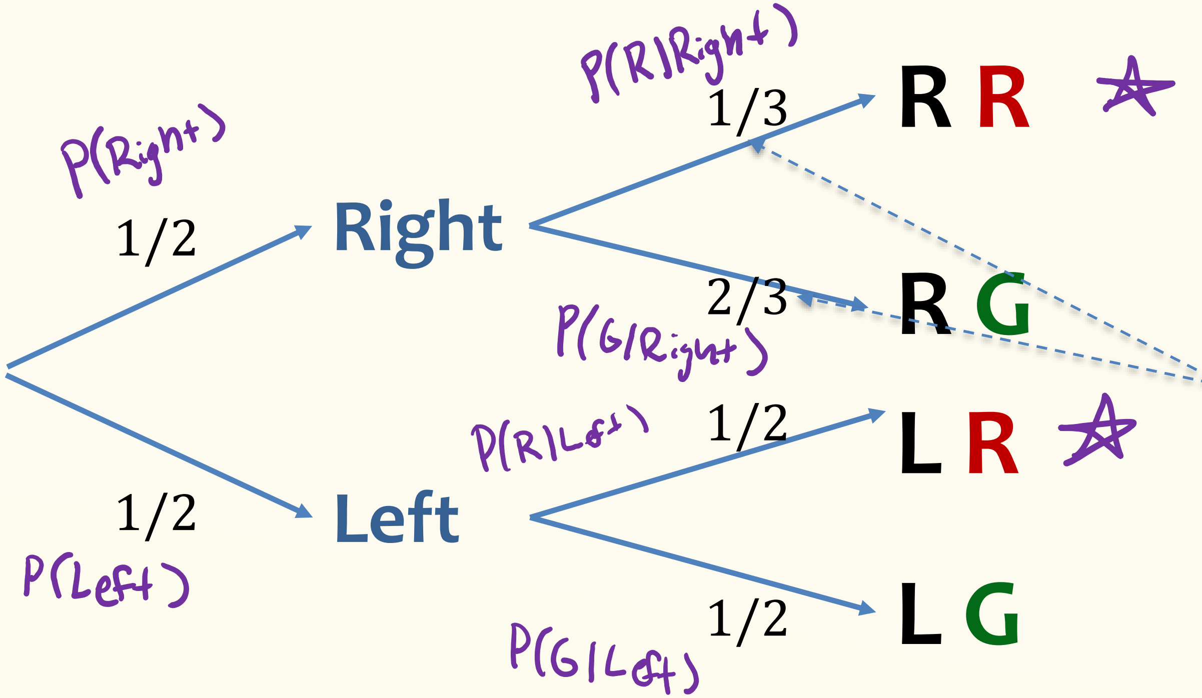
Alice picks a random ball from a random pocket.

[Both pockets equally likely, each ball equally likely.]

Pr(Red)

Sequential Process – Non-Uniform Case

Tree Diagram



- **Left pocket:** Two red, two green
- **Right pocket:** One red, two green.

1/3 = $\mathcal{P}(R \mid \mathrm{R})$ and 2/3 = $\mathcal{P}(G \mid \mathrm{R})$

$$\mathbb{P}(\mathbf{R}) = \mathbb{P}(\mathbf{R} \cap \mathbf{Left}) + \mathbb{P}(\mathbf{R} \cap \mathbf{Right})$$

(Law of total probability)

$$= \mathbb{P}(\mathbf{Left}) \times \mathbb{P}(\mathbf{R}|\mathbf{Left}) + \mathbb{P}(\mathbf{Right}) \times \mathbb{P}(\mathbf{R}|\mathbf{Right})$$

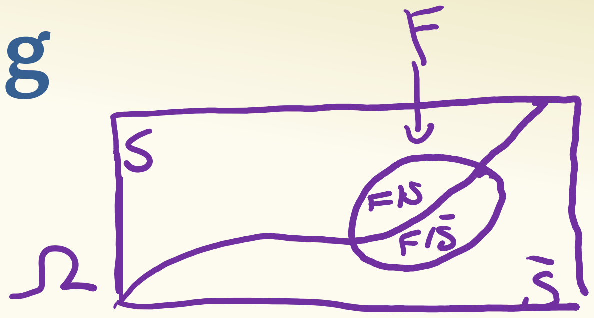
$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Our First Machine Learning Task: Spam Filtering

S = is spam, F = has "FREE"



Subject: “FREE $$$ CLICK HERE”

Goal: $P(S|F)$

$S, \bar{S}$ partition

What is the probability this email is spam, given the subject contains “FREE”?

Some useful stats:

- 10% of ham (i.e., not spam) emails contain the word “FREE” in the subject. $\Pr(F|\bar{S}) = 0.1$
- 70% of spam emails contain the word “FREE” in the subject. $\Pr(F|S) = 0.7$
- 80% of emails you receive are spam. $P(S) = 0.8$ $\quad P(\bar{S}) = 1 - 0.8 = 0.2$

$$P(S|F) = \frac{P(F|S)P(S)}{P(F)} = \frac{P(F|S)P(S)}{P(F|S)P(S) + P(F|\bar{S})P(\bar{S})} = \frac{0.7 \times 0.8}{0.7\times 0.8 + 0.1 \times 0.2} \approx 0.966$$

Bayes Theorem with Law of Total Probability

Bayes Theorem with LTP: Let $E_1, E_2, \ldots, E_n$ be a partition of the sample space, and F and event. Then,

$$P(E_1|F) = \frac{P(F|E_1)P(E_1)}{P(F)} = \frac{P(F|E_1)P(E_1)}{\sum_{i=1}^{n} P(F|E_i)P(E_i)}$$

Simple Partition: In particular, if E is an event with non-zero probability, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F|E)P(E) + P\left(F\middle|E^C\right)P(E^C)}$$

E, E^c partition

Agenda

- Conditional Probability
- Bayes Theorem
- Law of Total Probability
- Bayes Theorem + Law of Total Probability
- More Examples

Example – Zika Testing

Usually no or mild symptoms (rash); sometimes severe symptoms (paralysis).

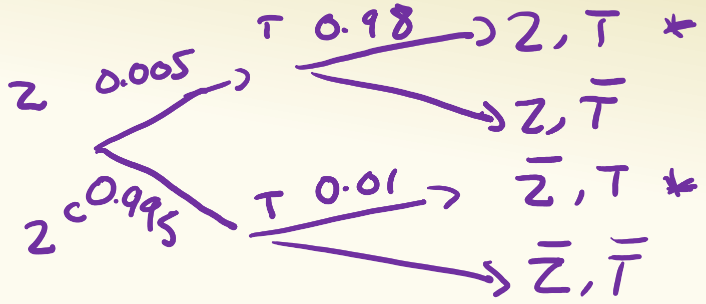
During pregnancy: may cause birth defects.

Suppose you took a Zika test, and it returns "positive", what is the likelihood that you actually have the disease?

- Tests for diseases are rarely 100% accurate.

This example and following slides are from Lisa Yan (Stanford).

Example – Zika Testing



Z = have Zika, T = test positive

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive") $P(T|Z) = 0.98$
- However, the test may yield a "false positive" 1% of the time $P(T|\bar{Z}) = 0.01$
- 0.5% of the US population has Zika. $P(Z) = 0.005$

What is the probability you have Zika (event *Z*) if you test positive (event *T*).

Goal: $P(Z|T)$

$$P(Z|T) = \frac{P(T|Z)P(Z)}{P(T)} = \frac{P(T|Z)P(Z)}{P(T|Z)P(Z) + P(T|\bar{Z})P(\bar{Z})} = \frac{0.98 \times 0.005}{0.98 \times 0.005 + 0.01 \times 0.995}$$

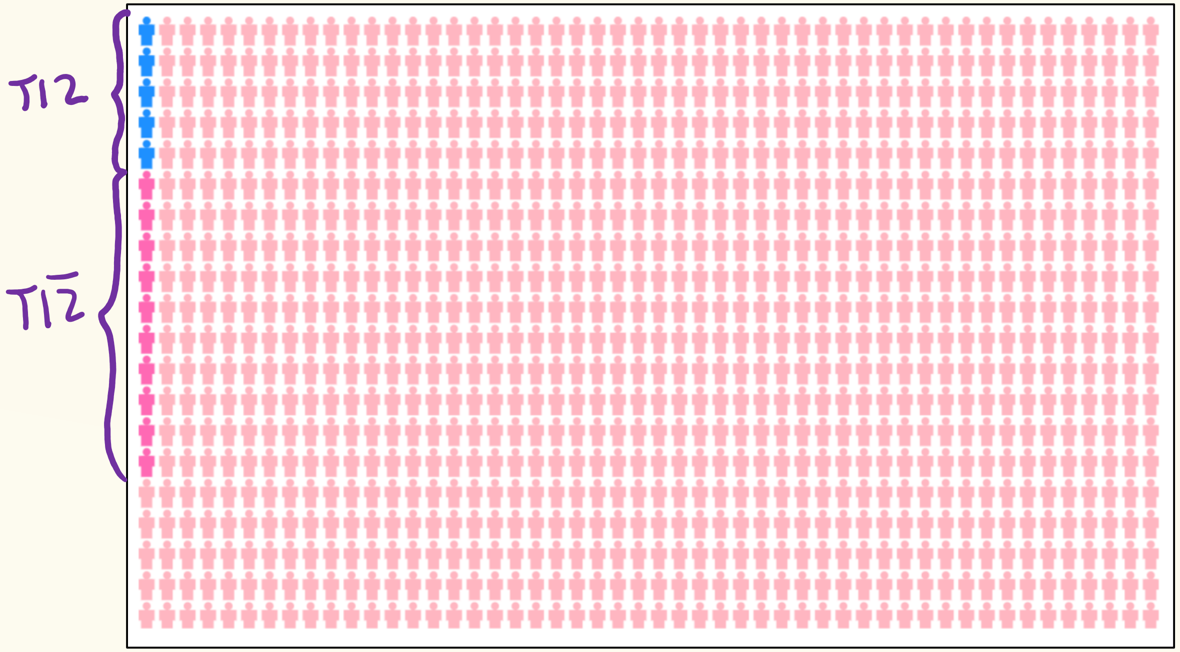
$$= 0.33$$

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika (“true positive”)
- However, the test may yield a “false positive” 1% of the time
- 0.5% of the US population has Zika.

What is the probability you have Zika (event *Z*) if you test positive (event *T*).



Suppose we had 1000 people:

- 5 **have Zika** and **test positive**
- 985 **do not have Zika** and **test negative**
- 10 **do not have Zika** and **test positive**

$$\frac{5}{5+10} = \frac{1}{3} \approx 0.33$$

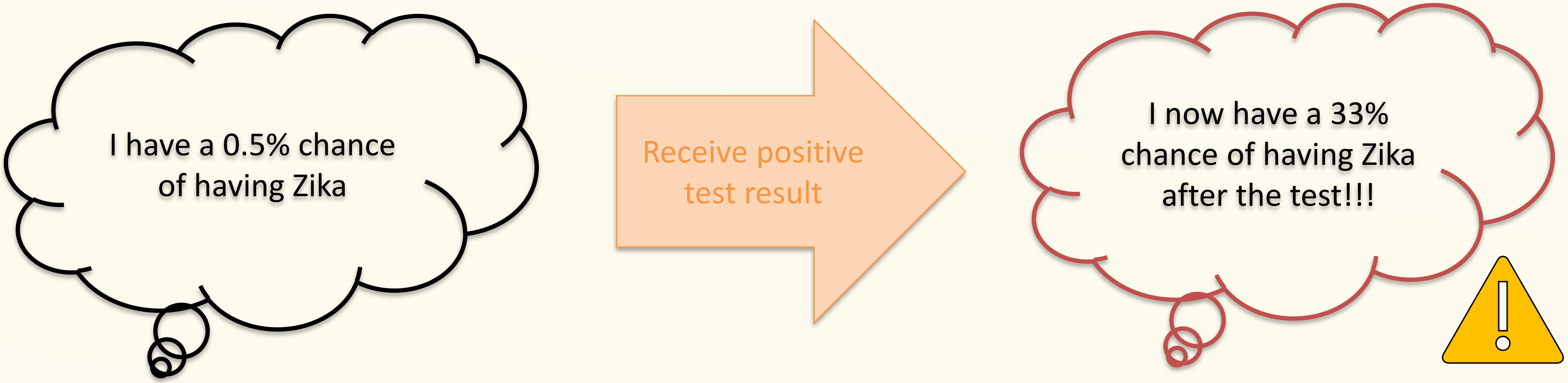
[Demo]

Philosophy – Updating Beliefs

While it's not 98% that you have the disease, your beliefs changed **drastically**

Z = you have Zika

T = you test positive for Zika



Prior: P(Z)

Posterior: P(Z|T)

Example – Zika Testing

Suppose we know the following Zika stats

- A test is 98% effective at detecting Zika ("true positive")
- However, the test may yield a "false positive" 1% of the time
- 0.5% of the US population has Zika.

What is the probability you test negative (event $\bar{T}$) if you have Zika (event Z)?

$P(\bar{T} | Z) = 1 - P(T | Z) = 0.02$

Proof

LTP

$P(A|B) + P(\bar{A}|B) = \frac{P(A \cap B)}{P(B)} + \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

$A, \bar{A}$ partition Ω

Conditional Probability Define a Probability Space

The probability conditioned on A follows the same properties as (unconditional) probability.

Example. $\mathbb{P}(\mathcal{B}^c|\mathcal{A}) = 1 - \mathbb{P}(\mathcal{B}|\mathcal{A})$

Formally. $(\Omega, \mathbb{P})$ is a probability space + $\mathbb{P}(\mathcal{A}) > 0$

➡ $(\mathcal{A}, \mathbb{P}(\cdot\,|\mathcal{A}))$ is a probability space